KONINKLIJKE VLAAMSE ACADEMIE VAN BELGIE
VOOR WETENSCHAPPEN EN KUNSTEN

MATH ART SUMMIT

28 06 2012

Ed.: Dirk Huylebrouck

CONTACTFORUM

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Contactforum "MATH ART SUMMIT"

CONTENTS

PART 1: TALKS HELD AT THE ROYAL FLEMISH ACADEMY OF BELGIUM, ON MAY 24 AND MAY 25

PRESENTATION AND OVERVIEW OF THE CONFERENCE CIRCUIT BRIDGES

Reza Sarhangi

Division of XXX, University XXX
** Institute for XXX*

1. INTRODUCTION

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MATHEMATICS WITHOUT WORDS NOR FORMULAS

Slavik Jablan and Ljiljana Radović*

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1. SYMMETRY

The laws of nature and the objects of human creation are representations of symmetry. A.V. Shubnikov refers to this in his book "Symmetry in Science and Art" [1]. He defines symmetry as "the law of construction of structural objects". Owing to its universality and synthesizing role in the whole scientific system, certain modern-day authors give to the theory of symmetry the status of a philosophy category, in terms of its ability to express the fundamental laws of order in nature. The existence of the word for symmetry in different languages: „*simmetria*" in the Greek and Latin, „*samita*" in Sanscrit, „*ketse*" or „*toam*" in Hebrew, „*dei cheng*" in Chinese, „*taisho*" or „*kensei*" in Japanese, witnesses about the importance and universality of the concept of symmetry. In European culture, the meaning of the word „symmetry" originates from Greek philosophy and aesthetics. The term „symmetry" is connected to a whole spectrum of philosophic-aesthetic terms: harmony, proportionality, well-behaved form, *etc*.

Symmetry in art reflects symmetry in nature. Since Paleolithic times, the oldest period of human civilization, symmetry has played an important role. A handprint was probably the oldest symbol in the history of mankind, the first attempt of a man to leave evidence of himself. He also made the negative of this image: he took a color and sprayed around his hand, leaving the part of the wall the hand occupied uncolored. After the first man made his handprint, others tried to do the same, and we obtained a pattern: an ornament, a structure based on repetition of the same motif (Fig. 1).

Figure 1: *Cave handprints*.

Many possibilities for new designs are based on antisymmetry („black-white“ symmetry). Our left and right hands are symmetric, but if you paint your left hand black, and right hand white, they become asymmetric. Antisymmetry is the symmetry of opposites: positive and negative, light and shadow, black and white. Its domain can be extended to different geometrical properties, e.g., the relations of „convex-concave“, or „over-under“. Therefore, antisymmetry can be used for so-called *dimensional transition.* If you have an antisymmetrical („black-white“) structure in the plane, e.g., antisymmetric rosettes, friezes, or ornaments, their black parts can be considered to be placed under the plane, with the white parts over the plane. In this way, from the 2-dimensional symmetry patterns of rosettes, friezes, or ornaments, we can obtain 3-dimensional symmetry structures with their invariant planes being tablets, bands, and layers.

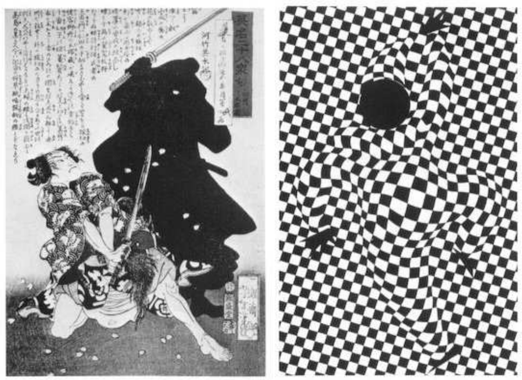
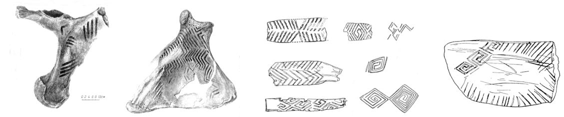


Figure 2: *(a) A man fighting with his shadow; (b) “Clown” by Victor Vasarely.*

This figure shows different forms of antisymmetry, the use of the relation „figure-ground“, „light-shadow“− we see a man fighting with his own shadow, and the form of a clown. The graphics by Victor Vasarely consists of black and white squares, which, thanks to the invisible borders between the squares, gives rise to the figure of a clown.

2. ORIGINS OF ORNAMENTAL ART

We have found that the oldest examples of ornamentation in Paleothic art were from Mezin (Ukraine), dated to 23 000 B.C. Note that 23 000 years is a time period five times longer than the complete written history of mankind. At first glance, the ornament on the right side of Fig. 3a appears to not be significant, it is an ordinary set of parallel lines. On the right side of Fig. 3b this pattern is transformed into a set of parallel zig-zag lines− an ornament with a symmetry group of type pmg, generated by an axis of reflection perpendicular to another axis of glide reflection. Let’s see how the creative process for the design of this ornament may have developed. Imagine a modern engineer who begins a construction project. At first he makes a rough sketch, and then he begins to work more seriously to solve the problem.
The next series of ornaments from Mezin is more advanced. The previously mentioned sets of parallel lines are arranged in friezes and meander patterns that can still be considered as sketches (Fig. 3c,d).



(a) (b) (c) (d)

Figure 3: *Basic patterns from Mezin.*

In Figure 4 we see the final result, the masterpieces of Paleolithic art─ the Birds of Mezin decorated by meander ornamentation and the bracelet. The man of prehistory has appled the symmetry constructions that he learned, and he has preserved them in bone for history. On the mammoth bone, he engraved meander patterns which represent the oldest example of a rectilinear spiral in the form of a meander (very popular, e.g., in the Greek art)

(a) (b) (c)

Figure 4: (*a) Bird of Mezin; (b) Mezin bracelet; (c) developed bracelet.*

In the Mezin bracelet we notice that there is a continuous transition from one ornament to another via a third ornament: on the left corner; you can see the meander ornamentation, then the set of parallel zig-zag lines used as a symbol of water, and again the continuous transition to another meander pattern. In order to make a continuous transition from one pattern to the next, it is neccessary to have a relatively high level of the mathematical knowledge and precision, which is unexpected for Paleolithic times [2]. A similar concept we can recognize in the „Metamorphoses“ by M.C.Escher (Fig. 5).

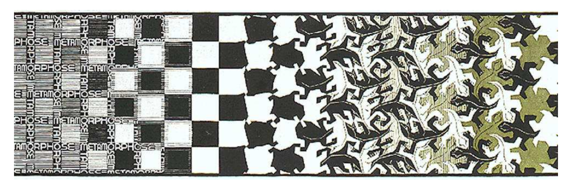


Figure 5: *“Metamorphoses” by M.C.Escher*.

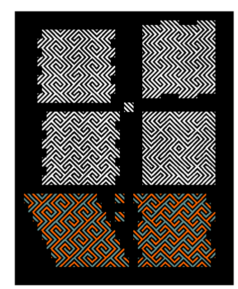
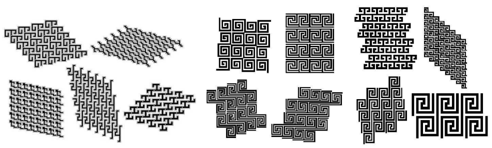


Figure 6: *Modular key-patterns*.

How is the continuous transition from one ornament to another made? The ornaments on Fig. 6 very differ one from another. Among them are black-white and colored ornaments, and at first glance, it appears that there is no unifying principle. Their common property is that they all consist of a single element (module). Notice the small black-white square in the middle. It consists of a set of parallel diagonal black and white strips. If this square is used as the basic motif, then all of these ornaments can be constructed from it. We call this method of construction *the principle of modularity*. Our goal is to construct all ornaments or structures by using the smallest number of basic elements (modules) and to obtain, by their recombination, as many different ornaments (structures) as possible. This module, a square or rectangle with a set of parallel diagonal black and white strips, we will call an Op-tile. It is the basis of Mezin meander patterns (Fig. 4).

Figure 7 shows a series of ornaments from Titsa culture (Hungary) and Vincha (Serbia), dating to 3 000- 4 000 B.C. They are painted on ceramic and can be found in all similar Neolithic settlements. How did the Neolithic people come to the idea of constructing such ornaments? We will try to show that all ornaments were derived from the simplest of human technologies: basketry, weaving, matting, plaiting, or textiles. Then the best of ornaments (in an aesthetic sense) were copied to the stronger media of bone, stone, and ceramics. Many of these ornaments are obtained from interlaced patterns (fabrics) or from textiles. If we take two bands of different colors and make the simplest possible interlacing pattern: „over-under“, „over-under“,...we obtain the antisymmetric („black-white“) checkerboard pattern. By replacing the simple code: „over-under“, „over-under“,...by a more sophisticated code (a repetitive algorithm), we obtain more complex and visually more interesting interlacing patterns [3]. Notice that ornaments from Vinca (Fig. 7b) are all based on meanders, continuing the tradition of Paleolithic ornaments from Mezin.



(a) (b)

Figure 7: *Neolithic ornaments from (a)Titsa culture (Hungary); (b) Vincha (Serbia).*

Figure 8: *Neolithic ornaments on ceramics: Tisza (Hungary), Cucuteni (Romania), Vincha (Serbia), Dimini (Grece), Tisza and Miskolc (Hungary), Serra d'Alto (Italy), Rakhmani (Greece).*

(a) (b)

Figure 9: *(a) Neolithic plate from Portugal; (b) Neolithic figurines from Vincha (Serbia) , Titsa region (Hungary), and Vadastra (Romania).*

In this sense, it is very convincing to observe that Fig. 9a shows a well dressed Neolithic man wearing a dress with stripes, similar to the module used for the construction of the ornaments from Mezin. In Fig. 9b there are Neolithic figurines, the first from Vincha, the second from the Titsa region (Hungary), and the third from Vadastra (Romania), where on parts of their clothes appear very similar types of meander ornamentation transferred from one culture to the other. This is also a testament to the notion that textile ornamentation was used as a model for similar ornamentation on ceramic. The best textile patterns were copied to ceramic vessels which requires great skill, since the surface of the ceramic vessels are curved. We can find similar examples all over the world (e.g., in Neolithic Lapita ceramics from Fiji (Fig. 10a), or Anasazi ceramics (Fig. 10b).

(a) (b)

Figure 10: *(a) Lapita ceramics (Fiji); (b) Anasazi ceramics.*

Usually, „black-white“ represents the relation between a figure and the ground. In almost every image we know which is figure and which is ground. However, if figure and ground are congruent, each of them can be called „figure“ or „ground“. Ornaments in which we are not able to distinguish a figure from the ground are called *key-patterns*, because their parts fit perfectly as a lock and key: the black part fits perfectly with the white and *vice versa*. These kinds of ornaments occur in different cultures distant in space and time: Chinese, Mexican, European, in Celtic ornaments (Fig. 11), and even in Neolithic ornaments from Fiji. Nearly the same ornaments occurred independently in the different parts of the world because thay have the same geometric basis: they are constructed by using the same geometric principles.

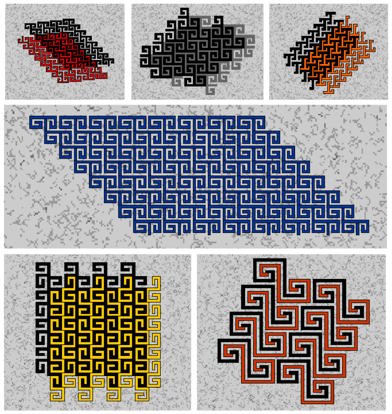


Figure 11: *Key patterns.*

3. MODULARITY IN ART

Modularity in science and art provides an application of the principle of economy in which from a minimal number of initial elements (modules), through their recombination, we construct the maximum number of possibilites. In this construction it is best to make an equal use of the modules. The tool useful for this is symmetry. Scientists have always searched for the basic building blocks of nature, in physics, chemistry, biology, and the other sciences. In Plato's philosophical treatise „Timeus", (written in the style of a Socratic dialogue), the four basic elements in nature: earth, fire, air, and water, are identified with the regular polyhedra: cube, octahedron, icosahedron, tetrahedron, and the fifth regular polyhedron, dodecahedron, represented the Universe. In physics, beginning from subatomic particles, atoms, or elementary energy entities, quarks, the units of matter or energy, scientists try to explain nature by using modularity. A similar tendency occurs in art and design (especially ornamental).

One of the most famous modular tiles are Truchet tiles, antisymmeric squares, used as the module for the constuction of all the ornaments shown in Fig. 12. The antisymmetric square or Truchet tile, was used beginning in Neolithic times, in the ornamental art of different cultures.

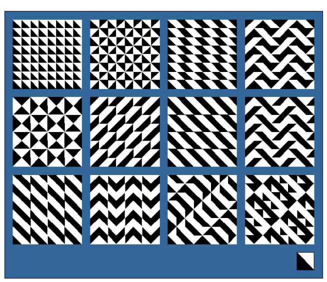


Figure 12: *Truchet tiles and patterns constructed from them.*

It is possible that there is no Neolithic culture that has not used it. An eclectic Dominican Father, Sebastian Truchet in 1704. started to analyze all possible arrangements (up to isometries) that can be obtained in a restricted part of the plane (in a square of dimensions 8×8) from this unit tile that was the „Truchet tile". Truchet's results, which represent the beginning of European combinatorics, are described in the work of Dominique Douat from 1722. A modification of the Truchet tile, that we will call the Kufic tile, is the simplest Op-tile, a white square with one black diagonal strip, or it's negative. It permits curvilinear variations and gives inexaustable possibilities for its use in graphical design and sculpture (http://www.iit.edu/~kufiblck/), especially for ornamental writing of texts in the form of so-called Kufic scripts, or Kufic calligraphy [10]. Truchet tiles, Kufic tiles and their curvilinear variations play a very important role in computer graphics [11]. From Kuffic tiles, Donald Knuth created very attractive TeX font [12].

"Knot Tiles" can be used as square playing cards to play an interesting game. The goal of the game is to close a knot (one-component curve). Every player at the beginning of the game gets the same number of randomly chosen cards. The player who closes a curve picks up all the cards belonging to the curve. The winner is the player who, at the end of the game, gets the most cards.

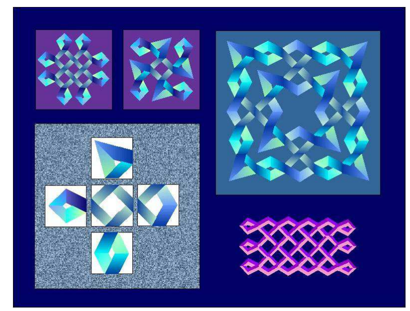


Figure 13: *Knot Tiles.*

The game has two variants: closed and open. In the first variant, players cannot see the cards of the other players. In the open variant, which is more interesting because all cards are open, every player can see the cards of the other players and, accordingly, can try to execute an optimal strategy. The mathematical optimization of the strategy of this game is very complex and probably exceeds the power of the modern computers. The element in the form of a kink is a kind of joker, because sometimes it closes a curve in only one step, but you need to use it very economically and try to use it at the optimal moment when you can close a larger curve and pick up many cards.

„Op-tiles" are a set of modular elements which consists of two rectilinear antisymmetrical tiles (a positive and negative). It can be extended to the set consisting of five elements, including two curvilinear tiles and one antisymmetrical Truchet-like tile. From these tiles it is possible to construct an infinite collection of plane black-white designs remaining us to the twentieth century art-style known as „Op-art" (optical art). Searching for the oldest example of „Op-tiles", we find them in the ornaments of Mezin (Ukraine, 23,000 B.C.). For such modular elements as a square with a set of parallel diagonal black and white stripes and its negative, we also use the name „Versitile", proposed by the architect Ben Nicholson who discovered the same family of modular tiles by analyzing Greek and Roman meander friezes and mazes. Key-patterns constructed from Op-tiles produce powerful visual effects of flickering and dazzle, thanks to the ambiguity which occurs due to the congruence between „figure“ (the black part) and „ground“ (white part) of the key-pattern, when our eye oscillates between two equally probable interpretations.

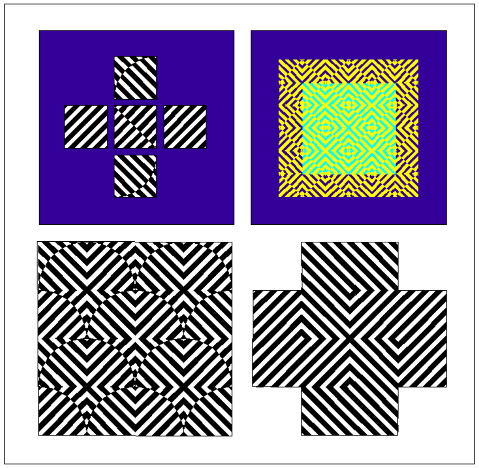


Figure 14: *“Op-tiles” and graphics constructed from them.*

The next figure shows mosaics from Antioch using the ambiguity of the relation „convex-concave“. In classical linear perspective where according to the dimensions of the objects (since all perspective lines converge to the same vanishing point), and light and shadow, we have no doubt which objects are convex, and which are concave. Here again we are faced with a dilemma to choose between two equally possible interpretations. If you look the figure in the middle, you are not able to decide whether it is convex or concave. The two figures on the left and right side of the figure contain Necker and Koffka cubes and offer the possibility of two equally valid interpretations.

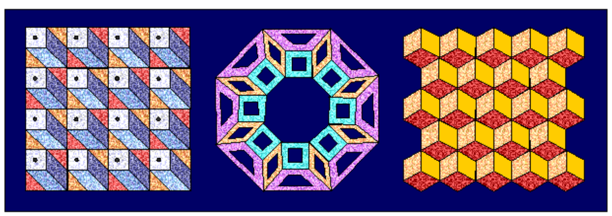


Figure 15: *Mosaics from Antioch.*

The basic building block of impossible objects is the Kofka cube: a regular hexagon divided into three congruent rhombuses that can be used as a space „turning point“ (switch). All three sides of the Kofka cube are identical, so we cannot tell from which of three equally possible points of view it is being viewed, whether it is convex or concave, or even if it represents a 3D-object, or is it a regular hexagon consisting of three rhombuses, which, acted upon by plane isometries, results in a rhombic tessellation. The oldest examples of impossible objects are based on the ambiguity of the relation „convex-concave“. However, impossible objects really entered into the domain of mathematics and the visual arts in the twentieth century.

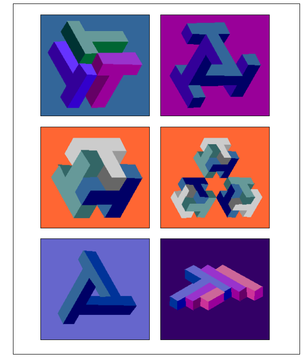


Figure 16: *Designs for a logo “3T”, using Koffka cubes.*

The Kofka cube and Penrose tribar are the first examples of that kind. They became very popular and are used as the geometric basis for the construction of many artworks based on impossible objects (e.g., graphics „Belvedere“, „Waterfall“, or „Ascending and Descending“ by M.C. Escher). As a result of combining (gluing) Koffka cubes we obtain different impossible figures. One Kofka cube produces the illusion „convex-concave“ or „2-dimensional-3-dimensional“, but the combination of two Koffka cubes with a common edge gives the very strange impression of an object with torsion. Continuing to piece together Kofka cubes, we obtain the most famous impossible object, the Penrose tribar (Fig. 17).

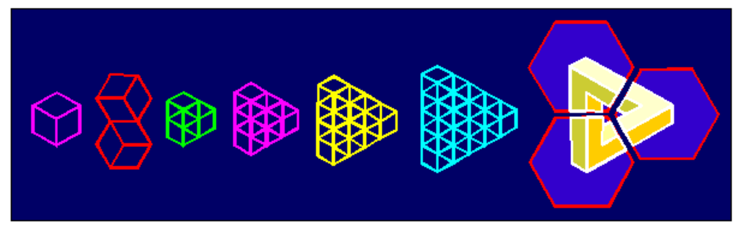


Figure 17: *A development from Koffka cube to Penrose tribar.*

4. ETHNOMATHEMATICS

Ethnomathematics is a relatively new field of the mathematics. The name „ethnomathematics“ is composed from the words „ethno“ and „mathematics“ and shows that different native, non-European cultures, developed their own mathematics and worked on it for centuries [4]. From examples of the ornaments from Tonga Island we can see that the people of this culture succeeded in exhausting almost all antisymmetry groups of friezes and to construct an enormous number of differents patterns, based mostly on antisymmetry. The most of these patterns can be constructed from only a few basic elements (modules). One of such modules is a Truchet tile.

Figure 18: *Tonga patterns.*

The name „mirror-curve", introduced by Paulus Gerdes, naturally follows from the following construction: take a regular rectangular grid RG[a,b] placed in a rectangle of the dimensions $a\times b$, with the external edges representing the mirrors, and with additional two-sided mirrors coinciding with the internal edges or perpendicular to them at their mid-points. A ray of light emitted from one mid-point, making a 45˚ angle with the edge, after a series of reflections, will close one component. Beginning at a different starting points, we continue with the same algorithm until the whole step-graph is exhausted. As a result, we obtain a mirror-curve. An analogous construction can be used to construct mirror-curves in an arbitrary portion of a plane polygonal tesellation. Different cultures constructed mirror-curves: knot diagrams placed in a plane tessellation. They occur in Tchokwe sand drawings, and Tamil and Celtic ornamental art. We are interested in understanding the mathematical principle behind their construction.

Begining with a portion of a polygonal tessellation of the plane, we connect the mid-points of adjacent edges and obtain a 4-valent graph: four new edges called *steps* meet at every vertex of this graph. We move along an edge of the graph until we come to a vertex; we continue our path by going straight through the edge laying between the two remaining edges. Every closed path obtained in this way, where every step occurs only once, is called a *component*. A mirror-curve is the set of all of its components. Every mirror-curve can be transformed to a knot or link diagram by introducing the relation „over-under" at every vertex of the graph. After introducing the relation „over-under" at every vertex, the vertices of a mirror-curve are called *crossings*. A "perfect" design (mirror-curve) has only one continuous curve (component). Tamil curves, consisting of a single curve are called „*pavitram*" („ring") or „*Brahma-mudi*" („Brahma's knot") and represented a kind of cultural ideal.

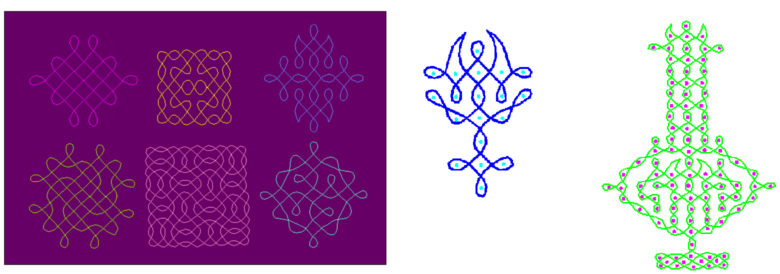


Figure 19: *Tamil mirror-curves.*

Tchokwe sand drawings, called „*sona*" designs played a very important role in the transmission of knowledge from one generation to the other. Children enjoyed drawing mirror-curves in the sand with their fingers, simple stylized drawings of different animals, birds, or human figures, followed by stories and tales. More complex drawings were carried out only by the experienced story tellers („*akwa kuta Sona*" = „those who know how to draw"), which played the role of highly valued teachers, part of the intellectual elite of the Tchokwe society.
Celtic knots are the highest order of interlaced ornaments, and they are also constructed as mirror-curves. From the mathematical point of view, mirror-curves belong to the knot theory and can be analyzed as knot and link diagrams: projections of knots and links onto a plane [8,9].

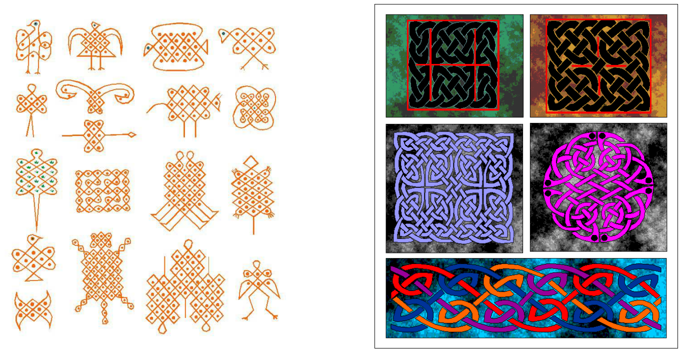


Figure 20: Tchokwe *sand drawings and Celtic knots.*

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DIAGRAMMATICS: ART, LANGUAGE, AND MATHEMATICS

Radmila Sazdanovic

Division of XXX, University XXX
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1. INTRODUCTION

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SYMMETRY SENSE

Jim Hausman

Division of XXX, University XXX
** Institute for XXX*

1. INTRODUCTION

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SYNERGIES IN ACTION! THE EXPERIENCE WORKSHOP MATHART MOVEMENT IN THE EXPERIENCE-CENTERED EDUCATION OF MATHEMATICS THROUGH ARTS, SCIENCES AND PLAYFUL ACTIVITIES

Kristóf Fenyvesi

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1. INTRODUCTION

The growing technologization, digitalization, network-ization and increasing complexity of everyday life making our surrounding reality highly mathematized. Through various channels mathematics is structuring our society, it has a fundamental impact to our culture and an enhanced significance for all citizens in the well-developed countries. Oddly enough, the abstractness of mathematics as a science makes it as a special discipline and detaches it from the context and experience of everyday life. This detachment often raises difficulties and negative attitudes towards mathematics. Mathematics, as a special discipline, also often understood as a male domain. The continuously ascending gap between the mathematics as a governing knowledge of our everyday life and the negative attitudes toward mathematics [1] holds innumerable dangers for the welfare societies. E. g. the lack of engineering students can dramatically hinder innovation. Just as the escalation of the non-understanding of the determining technical, economic, social processes also can lead to the rapid weakening of the equal access to the controlling systems which is the base of the democratic welfare state. Flexibility and proactive initiatives are needed to prevent the further growth of the 'math gap'. We have to discover, establish and verify new approaches that characteristically make clear the mathematics' diversified cultural embedding and reinforce its role and significance in society [2].

The knowledge gained through blurring the boundaries of art, science, and technology becomes our common experience of heterogeneity, and this common experience is also expressed in the transformation of our sociocultural practices. An integrated approach to art, science, technology, mathematics, and the flexibility of the learning environment that considers all these sociocultural changes can be productively and most efficiently established in the educational system. This should take place concurrently with acquiring new models of networked researching [3], learning through action and reflection, collective cooperation [4, 5], and experiential methods, which often infuse direct experience with the learning environment and content.

The school that is built upon the dogmatic segmentation of knowledge and the pedagogy of strictly fixed roles is less effective today. By now, multi-modal flexibility that involves several variables of the teaching / learning process has simultaneously become an indispensable precondition for meeting the ever-increasing and wide-ranging demands that education has to face today. In the field of experience-centered mathematics education, all these developments prompt us to enlarge the set of pedagogical tools and materials to complement the STEM (Science, Technology, Engineering and Mathematics) integration with aesthetic, artistic, creative and holistic design aspects and make the most of the successful models of cooperation among mathematics, sciences, and arts [6].

What do mathematicians mean by 'beauty' and 'creativity'? And for artists who make use of mathematical knowledge in their work, what constitutes a research practice? What happens to mathematical knowledge, discoveries, visualizations, models, and simulations when they become the subject of aesthetic reception, or artistic application, or performance? What mathematical significance can be attributed to a work of art or a game? What new results can be produced from the special perspective that combines mathematics, arts, and games? How can we make the most of these results in education?

In the open network of the Experience Workshop Math-Art Movement for the experience-centered education of mathematics, established in 2008 in Hungary and led by the author of this article, almost one hundred scholars, artists, engineers, architects, teachers of various subjects, craftsmen, and toymakers look for answers to these questions through various forms of interactive, hands-on, skill based, play-oriented, and experiential combinations of mathematics and arts. Our aim is to involve the students and their teachers and families into a vibrant dialogue between the mathematical and artistic points of view and raise our own personal interests in the field where mathematical and artistic thinking and practice merge. By researching the various possible connections between scientific and artistic education the Experience Workshop Movement's members are contributing to the development of new educational approaches that can be fruitfully implemented through the organization of math-art festivals, art and science workshops, interactive math-art exhibitions, conferences, and even in the development of new inside / outside curricula in everyday teaching [7].

The Experience Workshop contributes to several international projects. It was an official event of the European Year of Intercultural Dialogue in 2008, and of the European Year of Creativity and Innovation in 2009, and of the Pécs2010 - European Capital of Culture project in 2010. In 2011-2012 the Experience Workshop leads a Hungarian-Croatian cross-border co-operation project for the development of mathematics, science and art education in Hungarian and Croatian high schools. The members of the movement are cooperating with many local and world organizations, institutions, international research groups, university programs, and art communities which are in turn interested in the relationship between mathematics, art, science, games, and education. Nearly ten thousand students and several hundred teachers and parents have attended our events and our publications [8, 9] are becoming popular among a growing circle of experts in the Hungarian educational, artistic, and scientific discourses.

EXPERIENCE WORKSHOP
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www.experienceworkshop.hu

Figure 1: *The Experience Workshop's logo, designed by Bálint Rádóczy, combines a paradoxical geometric object, the Möbius strip, with a difficult mathematical idea, the infinite.*

The goals of the Experience Workshop:

a) Integrating the pedagogical results of using art, science, and play-centered learning into the teaching of mathematics in activity- and experience-centered educational programs.
b) Organizing various math-art events in Hungary and in its neighbouring countries for the introduction of best practices concerning the experience-centered teaching of mathematics.
c) Familiarizing the students and the current and future teachers in public education with the most recent results of experience-centered mathematics education; researching, collecting, and publishing the main domestic and international achievements and making these accessible for the broadest scientific, artistic, and teaching communities.
d) Expanding the set of tools used for increasing a learner's mathematical, logical, combinatorial, and spatial abilities, structured thinking skills, developing perception, aesthetic sensibility, motivating collaborative problem solving, interdisciplinary and inter-artistic approaches on all levels, and in every field of education.

2. THE IMPORTANCE OF USING TOOLS IN EXPERIENCE-CENTERED MATHEMATICS EDUCATION

Representatives of the most notable trends in reform pedagogy (e.g. Montessori, Steiner, Freinet, Petersen, Neill, Parkhurst) and the activity-, experience-, and game-centered alternative pedagogies that draw upon their work all attach great importance to the use of tools embedded in the teaching and learning process [10]. The educational purpose of the revival of traditional crafts, the play-oriented and creative activities involving various objects, the use of modelling kits, manipulatives, and multimedia teaching materials, creating artworks and innovatively re-organizing the learning environment together can create various opportunities. These opportunities should be given a key role in the math-art education practice: *"Mathematics, the language of science, is secured by the young child's understanding of basic symmetries: translation, rotation and reflection. These rudimentary transformations are used to develop a primitive topological space constituted by relations of proximity, separation, surrounding and order. These early mathematical relations are qualitative, not quantitative and their representations are structured symbolically; not as numbers. They are developed through work with 3-dimensional activities such as block play, woodworking and sculpture and expressed through the practice of art: drawing and painting."* [11]

It has been shown that a creatively used educational tool – which can be a simple piece of fruit, a game, a toy, a scientific model or a work of art – can to a considerable extent alter the relationship of students to their studies and even to their teacher and to each other: "*All these activities extend the standard teaching programs and develop the creative thinking of the students by burdening their left and right cerebral hemispheres more or less equally balanced, and by facilitating interaction between the two hemispheres of their brain. The creative artistic practice helps the children to understand and familiarize the algebraically formulated regularities of mathematics, and contributes to their abilities to make abstract mathematics conscious […]. Experience workshops mobilize synergies with a multidisciplinary approach and cooperative learning.*" [12]

A tool used in the educational process proves to be well chosen and successful in practice if the activity carried out with it redefines every actor's position in the pedagogical situation. Consider the following conditions: (a) if both the object and the pedagogical method used for its introduction into the process arouse curiosity and stimulate a playful mind, encouraging both the learner and the teacher to engage in discovery, participative analysis, experimentation, creative work or free play, and (b) if both the student and teacher are given the opportunity to change roles occasionally, then the entire structure of the teaching/learning process is flexibly transformed, making the accomplishment of the complex pedagogical objectives much easier.

By using activity-centered teaching methods in mathematics education, the traditional unidirectional communication models can be replaced by cooperative learning methods and content that focuses on the student and is sensitive to the various social and individual challenges. When students participating in the learning process expand their mathematical knowledge and develop their abilities and skills through activities of their own and activities carried out together with their peers, they have more of a need for self-directed learning. At the same time they acquire the complex personal and social abilities that are required for successful cooperation and collective problem solving. The interaction between individual and collective learning through artistic and playful contents pave the way towards a balanced approach enabling the mathematics teacher to act as a facilitator, to bring existing knowledge to the surface in subtle ways, and to create a nurturing atmosphere for learning from one another [13].

As it is often described in the literature on experiential education, these new roles and structures may seem unfamiliar to both students and adults in school: "*Traditionally, students have most often been rewarded for competing rather than cooperating with one another. Teachers are not often called upon for collaborative work either. Teaching has traditionally been an activity carried out in isolation from one's peers, behind closed doors. Principals, accustomed to the traditional hierarchical structure of schools, often do not know how to help their teachers constitute self-managed work teams or how to help teachers coach students to work in cooperative teams.*" [14] However, our experiences have shown that the Experience Workshop's events stimulate and make an important contribution to the natural transition from the old school to the experience- and student-centered new structure.

3. INTRODUCING THE EXPERIENCE-CENTERED APPROACH TO THE TEACHING OF MATHEMATICS IN HUNGARIAN PUBLIC EDUCATION

The connections between mathematics and the arts, the creative and practical application of hands-on activities and, last but not least, the teaching of mathematics using an interdisciplinary and inter-artistic approach have a rich modern tradition and an extensive international system of institutions. Oddly enough and contrary to wide distribution of the originally Hungarian mathematics education tools and mathematical toys (e. g. Dienes blocks, Rubik's Cube), all this is however rather under-represented in Hungarian public education and it is nearly entirely missing from the Hungarian teacher training. The introduction of the results of reform pedagogies and many of their attractive aspects to different public educational institutions in Hungary is often met with numerous difficulties [15]. The development or implementation of a complex pedagogical method in the teaching of mathematics might, in many cases, require the transformation of the entire institutional structure. Given that even a small change affecting teaching norms or the general structure of subjects may give rise to conflicts between the different actors in the education system, most of the pedagogical innovations in the teaching of mathematics can be introduced in public education only in the form of activities carried out outside the regular classes. However, in addition to introducing forms of education outside classes and working out programs designed to identify and develop talents, it is equally important to study and develop the modern methods of education within the school, spread best practices, and recognize and make the most of the simple fact that everybody has a special talent for something.

The state-controlled top-down transformation of the structure of subjects may not be the only way to resolve the conflict between the constraints it imposes on the education process and the interdisciplinary and inter-artistic foundation of mathematical knowledge. Respecting the present structures of Hungarian public education, we explore the interdisciplinary and inter-artistic connections in an experience-centered manner, thereby stimulating the students, their parents and teachers themselves, offering new opportunities for the teaching of specific topics of mathematics. All this requires the math teacher and sometimes the families to engage in research, and have intensive interest in issues of science, art, technology, culture and education that goes well beyond the narrow boundaries of the subject, and the acquisition of a certain level of expertise in the objects, technologies, methods and activities to be used.

4. THREE EXAMPLES: THE EXPERIENCE WORKSHOP'S GIGANTILE, THE GEOMETRICAL HOPSCOTCH AND SAXON'S POLY UNIVERSE TOY FAMILY

Squids-and-Rays. Robert Fathauer's revolutionary *Squids-and-Rays* puzzle, originally with small pieces that resemble sea creatures and fit together in an almost endless number of combinations, is a great set for educational use to carry out open-ended plays that encourages creativity. With the artist's leadership the Experience Workshop's facilitators enlarged the puzzle pieces and after the playful understanding of the basics of tessellations by tables (Fig. 2), a huge, spectacular 'GianTile Workshop' (Fig. 3) took place at the Hall of the Kaposvár University. In Fathauer's workshop, students explored tessellations using two different shapes of tiles, squids and rays. They learnt what a vertex is and how vertices can be used to characterize a set of tiles. They built the different types of vertices allowed by the *squids-and-rays* tiles. They also learnt the different types of symmetry possible in tessellations and constructed squid-and-ray tilings with each type of symmetry. Large tessellations with five-fold rotational symmetry were also built.

Figure 2-3: *Squids-and-Rays tessellations by Robert Fathauer at Experience Workshop's math-art festival at the Kaposvár University.*

Geometrical Hopscotch. The *Geometrical Hopscotch* is created by a painter Franciska Bali in the framework of the 'Ars GEometrica' course on math-art connections held by Csaba Hegyi DLA and Kristóf Fenyvesi at the Pécs University. The *Geometrical Hopscotch* is a five-piece series of large-scale models of the Platonic bodies. The sides of the three-dimensional bodies can be quickly folded out into the plane and the flat surface can be used as a special kind of hopscotch (Fig. 4-5). It is possible to write, draw or to stick number-, letter-, or picture-cards on the sides to explore all the logical, algorithmic interrelations which stem from the three-dimensional attributes of the bodies and the features of the planar objects. This way the traditional hopscotch game is extendable with a number of individual or cooperative cognitive games which highly support the education processes in the topic of changing between plane and space. The *Geometrical Hopscotch* can just as well be used outdoors.

Figure 4-5: *Ildikó Szabó's Geometrical Hopscotch Workshop for elementary (left) and secondary (right) school students.*

Poly universe toy family. The poly universe toy family (www.poly-universe.com) by the painter jános szász saxon is a toy for developing geometrical skills (fig. 6) and an artistic and mathematical form system (fig. 7) based on scale-shifting symmetry. These colourful geometric shapes are not only designed to aid colour and shape recognition - and to teach the solving of logical puzzles - but also offer the chance to play freely, learning through an artistic game or activity. Progressive use of colour and shape groups, and the encouragement of manual activity and reflective thought, create a constant challenge for the children. This maintains their desire for exploration, producing a continuous feeling of success. Having a direct, tactile connection with the geometric shapes develops children's sense of vision and touch. Through the recognition and discovery of correlations and linkages, cognitive and abstraction skills are improved. Following compositional play with the forms (fig. 5), incorporating key aspects of geometrical composition and art, children make their own geometric artwork from paper and organise an exhibition in the classroom (fig. 8). The poly universe toy family develops key skills in examining geometric shapes, proportions, symmetry, linkage points, directions, and colour combinations. It expands the limits of form and colour composition/combination. It is interesting to place poly universe shapes amongst nature (fig. 7) – e. g: as a colourful field of flowers to educate students about the relationship between maths and the natural environment. Such experiences help children develop knowledge, finding creative and imaginative solutions to problems that cross between the abstract and real-world.

Figure 6-7-8-9: the various uses of the poly universe toy family by the painter János Szász Saxon and the art critic by Zsuzsa Dárdai.

5. THE TRAVELING EXHIBITION OF THE EXPERIENCE WORKSHOP AND THE ARS GEOMETRICA ART – SCIENCE – EDUCATION GALLERY

The International Traveling Exhibition of the Experience Workshop was established in 2010 and funded by the donations of the participants of Bridges Pécs 2010 World Conference's Grand Exhibition. Our constantly growing mathematical and artistic collection includes nearly 80 pieces by artists and scholars from all over the world. These artworks are key pieces in the events organized in public schools, universities all around Hungary by the Experience Workshop. They can be used to illustrate the cultural, artistic, architectural and interdisciplinary foundations of mathematical thinking in many different ways.

The promotion and popularization of research activities and the publication of scientific and artistic results for a wider domestic and international audience is essential for the successful management of any higher education institution. Recognizing this need, we began to prepare a course in Science and Art Management in September 2011 at the Eszterházy Károly College of Eger, Hungary, providing a new platform for the training of science and mathematics teachers. Through several examples of internationally recognized initiatives, the course gives students an insight into the professional background of promoting science and art connections as well as giving them an opportunity to test, to try out, and to put into practice the knowledge and skills they have acquired. Practical activities can be conducted in our exhibition space set up at the College, which has been operating as an experimental math-art gallery since September 2011. The workshop gallery, whose unique themes and concept are reflected in its name and in its slogan *Ars GEometrica Gallery: Interactions and Border-Crossings in Art and Science*, functions as a completely new platform in the Hungarian mathematics teacher education. Here the students not only can learn about the best examples of promoting and popularizing science and contemporary art but can also gain professional experience while testing their abilities in special fields of their own interest such as: organizing exhibitions, symposia, and Art&Science Café sessions, the fundamentals of exhibition and education technology, scientific and artistic communication, project management, writing tender applications, learning fund-raising techniques, PR- and media management, web design, presentation techniques, and so on. We organize two to three exhibitions in the Gallery every semester. The opening events of the exhibitions and the Art&Science Café sessions organized during the period of the exhibitions are listed among the highly recognized cultural programs of the city of Eger. Moreover, the scientific symposia related to each exhibition are designed to strengthen the international professional reputation of the Eszterházy College. Together the opening events and the Art&Science Café sessions and the website of the Gallery [17] an excellent platform for popularizing all these contents, the innovativeness of the Eger College together with the famous wine culture of Eger, thereby recognizing the importance of connecting local interests in a global cultural space. The annually organized events of the Gallery will be summarized in the Ars GEometrica Almanac, a publication which will contain artistic and scientific documents produced as a result of the exhibitions, while multimedia material is accessible on the Gallery's website.

6. CONCLUSION

As William Byers ingeniously exposed in his book, entitled *How Mathematicians Think*, mathematicians make use of "ambiguity, contradiction, and paradox to create mathematics" [17]. Understanding Byers' insight in a fundamental way can open new perspectives in the mathematics education as well: *"Mathematics educators investigate mathematics as it is learned and taught. Therefore they are forced to consider not only the formal, objective aspects of mathematics but also the human dimension of the subject. They are forced to confront such questions as 'What is meaning?' 'What is understanding?' The result has been that various mathematics educators have developed a rather sophisticated approach to the nature of mathematics. These approaches have in common with my own a desire to free mathematics from an entirely 'objectivist' point of view, 'objectivist' in the sense that the meaning of mathematics is 'out there' in a mind-independent reality."* [17, p. 65] According to the methodical framework of the Experience Workshop, by the study, interpretation and creation of artworks a lot of ambiguous and creative aspects, possibilities of mathematics can be brought to light. Just like many artistic creations can be understood better by studying, interpreting or re-creating the 'mathematical element' in them. As an addition to the interactive activities offered by the Experience Workshop, the exhibition also shows how artistic-scientific exhibitions and workshops can be connected to schools, universities, festival events, and even art galleries.

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SLICING SURFACES WITH MATHEMATICA

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1. INTRODUCTION

From the second half until the end of the XIX century there was a movement in Europe and around the world in which most mathematicians started building models of surfaces which they used with their students in their classes. We must keep in mind that in that time there were no computers that professors could use with their students. So, in some cases, it was very difficult to visualize surfaces. The professor had to draw brilliantly. For these reasons, professors had to develop new techniques to explain surfaces to students, such as building models.

This movement in Europe had its apogee in Germany, where the mathematicians Felix Klein (1849-1925) and Alexander von Brill (1842-1935) started building models of surfaces that were unknown before then. After some years, Alexander von Brill's brother set up a company to sell models which some of them had been designed by his brother and Klein. In 1899 he sold the company and the new company was renamed as Martin Schilling.

In 1903 this new company published a catalogue which included 23 series of models and around 300 models of surfaces. After this, in 1912, the company published the last catalogue with 40 series and around 400 models, where some of them where designed by students of Klein and von Brill.

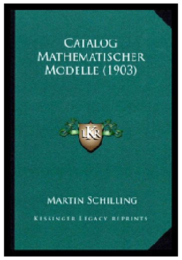

Figure 1: *Catalog Mathematischer Modelle (1903)*

This company sold models made in different materials such as plaster, wire and paper. But in this paper we are going to focus on the last type of models.

2. JOHN SHARP'S MODELS

Recently John Sharp published a book ([5], [6]) where he gives a method to build surfaces by intersecting them with two families of perpendicular planes. For example, if we want to build a cone of height 1, this method looks like this:

We take the equation of the cone and we calculate the intersection of it with two families of perpendicular planes which are parallel to the coordinate planes. These planes have the form $y = k$ and $x = k, k \in \mathbb{R}$. For each value of *k*, we obtain a curve, which in this particular case is a hyperbola. Keeping the same distance between each consecutive value of k (for example $k = 0.2, 0.4, 0.6, \ldots$) we obtain two families of curves.

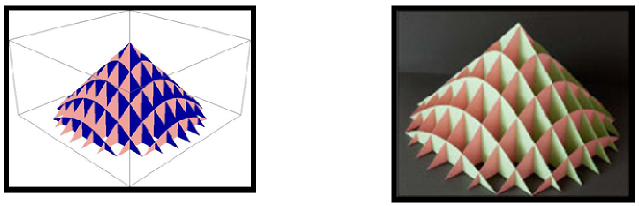

Figure 2: *Virtual and real model of the cone*

To finish the model we just need to make the slots in each piece. In the pieces of the first family of planes the slots goes from the bottom until the middle and in the other family from the middle to the top of the piece (see figure 3).

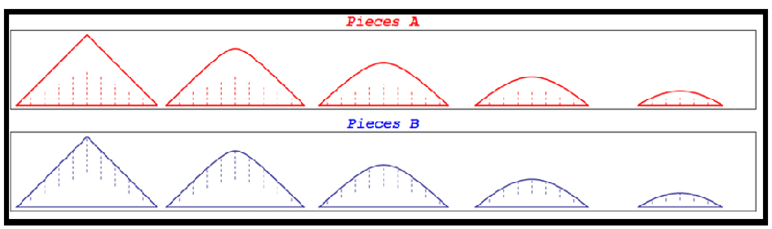

Figure 3: *Pieces of the cone*

But this method has one restriction; the intersection with the planes must pass through all the maximum and minimum points of the surface. Otherwise we would lose information. For example, in the case of the cone, if we do not have any plane along the vertex when we build the model, we will end up with a flat surface on the top instead of the cone (see figure 4).

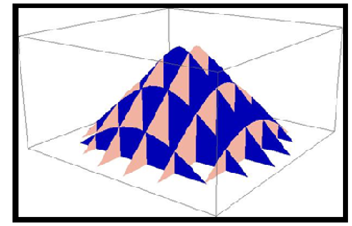

Figure 4: *Cone without the vertex*

Using this method we have built some new models, some of them based on surfaces that can be seen in the Imaginary Exhibition. Here we present some of them:

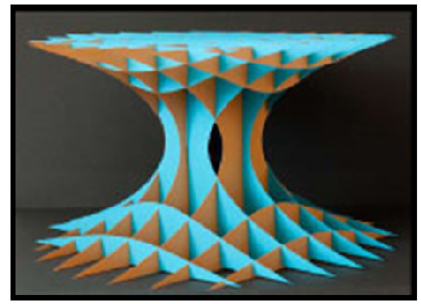

Figure 5: *Catenoid*

Figure 6: *Croissant*

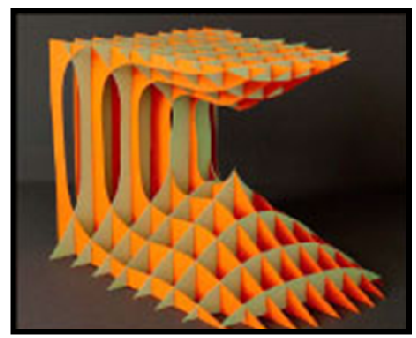

Figure 7: *Vis à Vis*

Figure 8: *Zeck*

3. NEW SLICED SURFACES

After using this method, we thought about using other families of planes which intersection with the surface could provide us a visualization of a geometric property of it, for example, the fact that the surface has a rotational symmetry. In the case of the torus, if we intersect the torus with planes parallels to the coordinate planes, we can build the model, but this has not any special property.

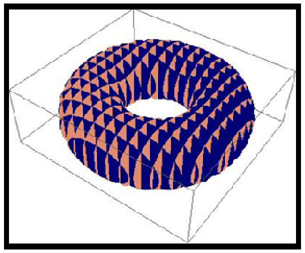

Figure 9: *Torus intersected with two family of perpendicular planes*

In [3] we explain how to build a torus with Mathematica by using as pieces a particular curve in the torus, the Villarceau circles. In the cited reference you can see all the details of the construction.

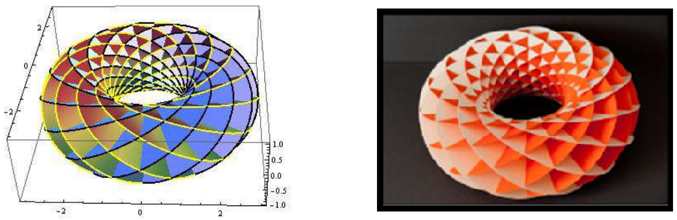

Figure 10*: Virtual and real model of the torus with Villarceau circles*

Using this idea we have designed new models which are described in the following subsections.

3.1 SLICED CYLINDER

The intersection of a right circular cylinder with a plane which is not horizontal gives always an ellipse (see figure 11).

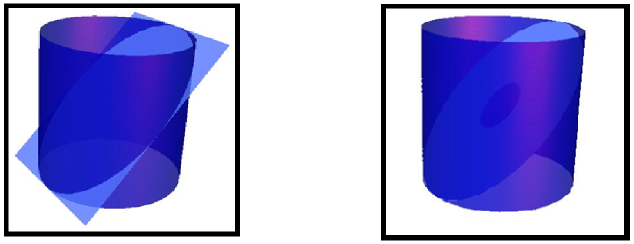

Figure 11: *A cylinder intersected by a plane (left) and the intersection curve (right)*

Since the pieces should not intersect with the rotation axes, we have to divide the ellipse in two parts for building the cylinder. If we rotate only one of the parts around the z-axes in one direction it does not generate the cylinder. Also, the same for the other part in the other direction, as it is showed in the following figure:

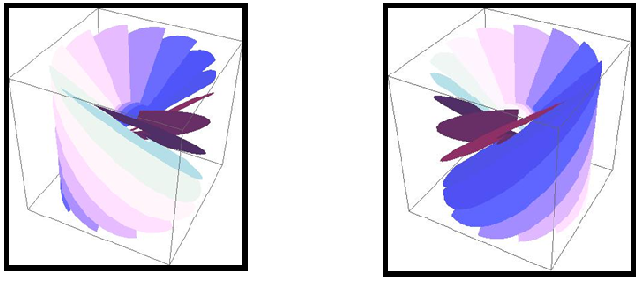

Figure 12: *Rotating the pieces around the z-axes*

But if we rotate both pieces at the same time, then they generate a cylinder:

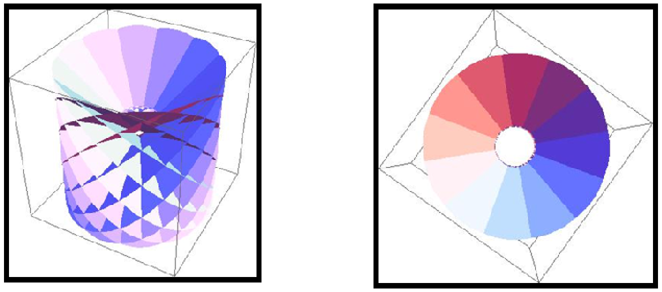

Figure 13*: Two images of the final model. The right one from a zenithal point of view*

In the last step, we have to calculate the slots of the pieces. In this case, for building this model we have used 24 pieces (12 copies A and 12 copies B) which have all of them the same shape.

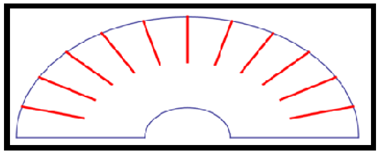

Figure 14: *Pieces A*

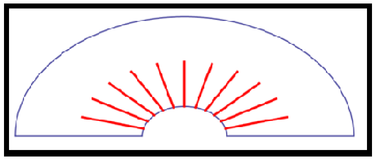

Figure 15: *Pieces B*

This model shows that the cylinder is a surface of revolution.

3.2 SLICED HYPERBOLOID

The intersection of a hyperboloid of one sheet with a plane which is not horizontal gives two lines (see figure 16).

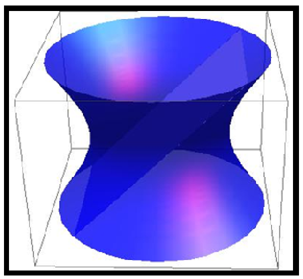

Figure 16: *Hyperboloid of one sheet intersected by a plane*

To build the hyperboloid, we have to divide the interior plane in two parts. If we rotate only one them around the z-axes it does not generate the surface. Also, the same for the other part rotated in the opposite direction:

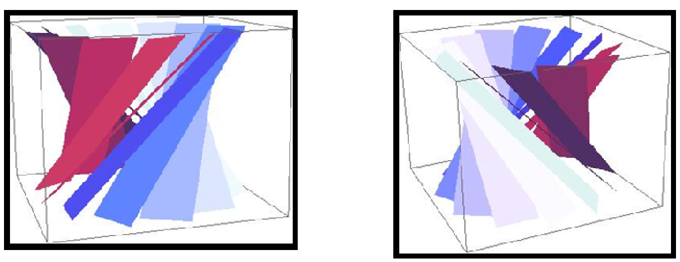

Figure 17: *Rotating the pieces around the z-axes*

But if we rotate both pieces at the same time, then they generate our surface:

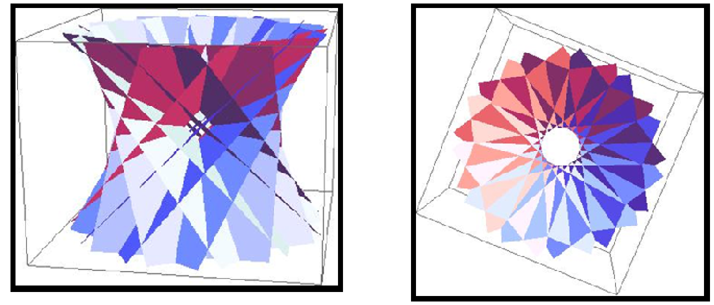

Figure 18: *Two images of the final model. The right one from a zenithal point of view*

In the last step, we have to calculate the slots of the pieces. In this case, for building this model we have used 24 pieces (12 copies A and 12 copies B) which have all of them the same shape.

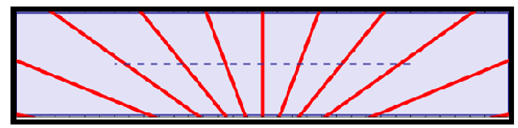

Figure 19: *Pieces for building the hyperboloid of one sheet*

In one of the families the slots goes from the bottom until the middle and in the other family from the middle to the top of the piece.

In this case, this construction shows that the hyperboloid of one sheet is a ruled surface and also, that it is a surface of revolution.

4. NOTES

I would like to thank Juan Monterde, from the University of Valencia, for his useful help with Mathematica and the design of some of these models. Also, I would like to thank Pilar Moreno, who realized some of the pictures.

If you are interested in the pieces for building any of the models or you want the pieces in other format (such as .pdf), please send me an e-mail.

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A BRIDGE BETWEEN ALGEBRA AND GEOMETRY *VIA* PROPORTIONS

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1. ABSTRACT

This paper is focused on the geometric interpretation of some algebraic relations defining metallic and plastic numbers. These numbers are considered in Architecture and Art as system of measurement and proportions started by dividing a segment in two parts. The variability of an integer coefficient, connected with the measure of the two parts, brings to different algebraic equations. The real positive solution of several second or third degree equations is a metallic or plastic number respectively. Among these, the first plastic number plays in 3D the well-known important aesthetic role of the golden number in 2D. The paper provides also some geometrical interpretations in terms of aesthetic proportions related to architectural projects, in particular to two different vaults. The natural correspondences between geometric entities and algebraic formulas bring us to consider this work as a bridge between Algebra and Geometry.

2. INTRODUCTION

Symmetry and proportions are always related to aesthetic aspects. The harmony of geometric forms involves architects, industrial designers, artists and inspires their creativity. We can think that each harmonic construction does not come from a casual chance but it is the natural synthesis between creativity, technical aspects and geometric knowledge.

In the first part of this paper the geometric interpretations of algebraic equalities, connected to second and third degree polynomial equations, are proposed. The solutions of these equations are related to static or dynamic proportions therefore we suggest recognizing geometric forms having inside these proportions.

In the second part of the paper we emphasize classical proportions in some architectural elements. We describe in particular an unusual and little known vault: the *Vela Quadrabile Fiorentina* designed by Vincenzo Viviani (1622-1703) at the end of the seventeenth-century [12]. This vault reveals many interesting properties related to static proportions.

The paper is organized as it follows. Section 2 is devoted to quadratic equations with solutions belonging to the Metallic Means Family. In Section 3 some 2D or 3D interpretations of algebraic equalities connected with static and dynamic proportions are presented.

In Section 4 the plastic number is related to 3D interpretations and to the projects of the Dutch architect D.H. Van Der Laan (1904 -1991).

In Section 5 and 6 the unusual *Vela Quadrabile Fiorentina* and the classical Sail Vault are investigated in relation to measures of parts. Static and dynamic proportions come out again in easy way.

3. THE MOST FREQUENT PROPORTIONS

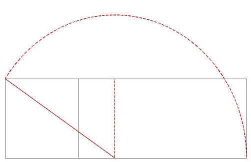
The proportion is the ratio between two geometrical elements, or between their respective measures (fixed the unit of measure).
In particular four segments A,B,C,D with lengths a,b,c,d respectively, are in proportion if $\frac{A}{B}=\frac{C}{D}$ or equivalently $\frac{a}{b}=\frac{c}{d}$, where $a,b,c,d \in R, b \neq 0, d \neq 0$

The *proportion* is said

- *rational* or *static* if $\frac{a}{b}$ is a rational number.

- *Irrational* or *dynamic* if $\frac{a}{b}$ is an irrational number.

Examples of *static proportions* are: *square* if $\frac{a}{b}=1$; *double (copper)* if $\frac{a}{b}=2$; *sesquialtera*[1] if $\frac{a}{b}=\frac{3}{2}$, *sesquitertia* if $\frac{a}{b}=\frac{4}{3}$; *pentatertia* if $\frac{a}{b}=\frac{5}{3}$……… Examples of *dynamic proportions*: *square root of two* if $\frac{a}{b}=\sqrt{2}$; *square root of three* if $\frac{a}{b}=\sqrt{3}$; *Cordovan* if $\frac{a}{b}=\frac{1}{\sqrt{2-\sqrt{2}}}$; *gold* if $\frac{a}{b}=\frac{1+\sqrt{5}}{2}$; *silver* if $\frac{a}{b}=1+\sqrt{2}$, *bronze* if $\frac{a}{b}=\frac{3+\sqrt{13}}{2}$, *nickel* if $\frac{a}{b}=\frac{1+\sqrt{13}}{2}$ …



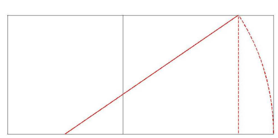


Figure 1: Construction of a bronze rectangle **Figure 2**: Construction of a nickel rectangle

[1] From Latin *sesqui* contraction of *semisque*

The last four numbers belong to the Metallic Means Family [3], which is composed of the positive solutions $x_1=\frac{p+\sqrt{p^2+4q}}{2}$ of the quadratic equations

(1) $$x^2-px-q=0 \quad (p,q \in N, \quad N=\{1,2,3....\}).$$

- Analysing the case $q=1$, the equation (1) gives for example the *golden number* if $p=1$, the *silver number* if $p=2$, the *bronze number* if $p=3$. In this case the product of the two solutions x_1, x_2 is $x_1x_2=-1$, consequently $x_2=-\frac{1}{x_1}$ and $x_1+x_2=x_1-\frac{1}{x_1}=p$. We can conclude that the decimal representations of x_1 and x_2 have the same decimal part.
- When $p=1$, the equation $x^2-x-q=0$ gives again the *golden number* if $q=1$, the *copper number* if $q=2$, the *nickel number* if $q=3$ and so on.

The equation gives integer roots, $x_1=n$ (and $x_2=-n+1$) if and only if $q= n(n-1)$, $n \geq 2$.

4. ABOUT THE QUADRATIC EQUATIONS

In this section we propose possible geometric interpretations of some algebraic equalities associated with the equation (1).
The choice $q=1$
Let us divide a segment in two parts of length a, b respectively ($a>b$) so that the following proportion is respected:

(2) $\frac{a}{b}=\frac{pa+b}{a}$ ($p \in N$).

From (2) we obtain

(3) $\left(\frac{a}{b}\right)^2=p\left(\frac{a}{b}\right)+1$

that is $\frac{a}{b}$ is solution of the quadratic equation

(4) $x^2-px-1=0$.

If a, b are the lengths of the rectangle R sides, R is a *golden rectangle* when $p=1$, R is a *silver rectangle* when $p=2$, and R is a *bronze rectangle* when $p=3$ (Figure 1).
Since the relation (3) is equivalent to

(5) $a^2-b^2=pab$

we can give a simple geometric interpretation of (5): the difference in area between two squares, having sides a and b respectively, is *p*-times the area of the rectangle with sides a and b.
Other interesting geometric interpretations can be deduced from (5):
If we multiply by π we obtain

$$\pi a^2-\pi b^2=p(\pi ab)$$

which means that the difference in area of two circles, having radius a and b respectively, is equivalent to p-times the area of the ellipse with semi-axes a and b ($p=2$ in Figure 3).

Let us introduce the Gerono's lemniscate, a regular 2D-curve [10] represented by the following parametric equations

$$\begin{cases} x = \ell\, sent \\ y = \ell\, sent \cos t \end{cases} \quad t \in [0,2\pi], \quad \ell > 0$$

where the real constant ℓ is the highest distance of the curve points from the double one (two Gerono's lemniscates with different choice of ℓ in Figure 4).

The area of the region bounded by the Gerono's lemniscate is given by $\frac{4}{3}\ell^2$.

Multiplying the relation (5) by $4/3$ we obtain

(6) $\frac{4}{3}a^2 - \frac{4}{3}b^2 = \frac{4}{3}pab$

The relation (6) shows that the area bounded by two Gerono's lemniscates of constant a and b respectively and the area of the rectangle with sides a and b are in *static proportion* $\frac{4}{3}p$ ($p=1$ in Figure 4)[2].

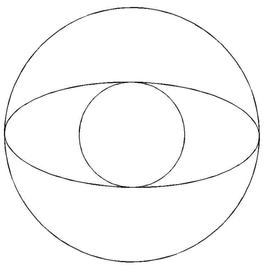


Figure 3

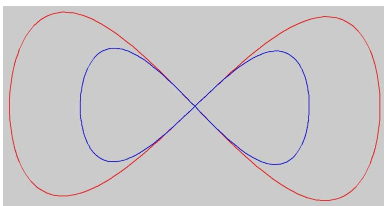


Figure 4: *Two Gerono's lemniscates*

Let us multiply the relation (5) by 4π to obtain

$$4\pi a^2 - 4\pi b^2 = 4p\pi ab$$

that is the difference in area of two spherical surfaces with radius a and b is $4p$-times the area of the ellipse having semi-axes a and b ($p=3$ in Figure 5).

Multiplying the relation (5) by $4\pi^2$ and writing it in the form

$$\pi(4\pi a^2 - 4\pi b^2) = p\, 4\pi^2 ab$$

we can say that two spherical surfaces with radius a and b have difference in area in dynamic proportion p/π with the area of the torus surface, obtained by revolving around the y-axis the circle $(x-a)^2 + y^2 = b^2$.

The choice $p=1$

Let us divide a segment in two parts of length a, b respectively ($a > b$) satisfying the proportion:

(7) $\frac{a}{b} = \frac{a+qb}{a}$ ($q \in N$).

The relation (7) is equivalent to

(8) $\left(\frac{a}{b}\right)^2 = \left(\frac{a}{b}\right) + q$

i.e. $\frac{a}{b}$ is solution of the quadratic equation

(9) $x^2 - x - q = 0$.

Considering a, b as side lengths of the rectangle R, then R is again a *golden rectangle* if $q=1$, R is a *copper rectangle* if $q=2$ and R is a *nickel rectangle* if $q=3$ (Figure 2).

As (8) is equivalent to

(10) $a^2 - ab = qb^2$,

the most simple geometric interpretation is that the difference in area of the square with side a and the rectangle with sides a and b, is q-times the area of the square with side b.

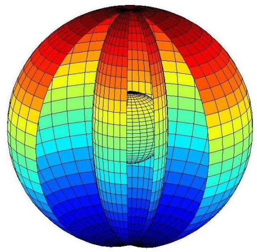


Figure 5

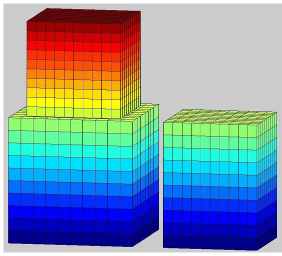


Figure 6

Multiplying by π the two members of (10) we obtain

$$\pi a^2 - \pi ab = q\pi b^2$$

that is the difference in area of the circle with radius a and the elliptical region with semi-axes a, b is in static proportion q with the area of the circle with radius b.

The relation (10) can be easily treated as the relation (5) in Section 2.1.

5. ABOUT THE CUBIC EQUATIONS

The first case. Let us divide a segment in two parts of length a, b respectively ($a > b$) so that the following proportion is respected:

(11) $\frac{a^2}{b^2} = \frac{pa+b}{a}$ ($p \in N$),

equivalent to

(12) $\left(\frac{a}{b}\right)^3 = p\left(\frac{a}{b}\right) + 1$.

Consequently $\frac{a}{b}$ is solution of the cubic equation

(13) $x^3 - px - 1 = 0$.

[2] All the graphic illustrations are generated by MATLAB®

which has always at least one real positive solution.
The choice $p=1$ gives the so-called *plastic equation*: its unique real solution is given by the *plastic number*, obtained for example by the Cardano's formula:

$$\Psi = \sqrt[3]{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{1}{27}}} + \sqrt[3]{\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{1}{27}}} \cong 1.32472...$$

For all life long the Dutch architect Dom Hans Van Der Laan (1904-1991) looked for a new system of measures and proportions, finding the golden section mainly related to 2D case [9, 14, 15, 18].
He focused his investigations on the plastic number that reveals important geometric properties in 3D and it looks like a natural extension of the golden number Φ [1, 5].

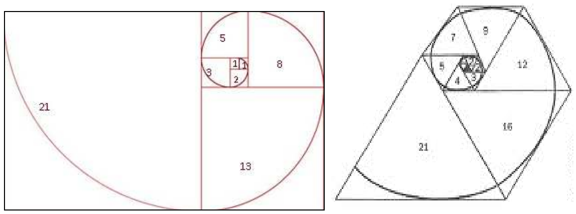


Figure 7: *Construction of the golden and plastic spirals*

In particular the plastic number, as the golden one, is connected to the following recurrence sequence, called Padovan sequence[3]:

$$P_{n+3} = P_{n+1} + P_n \quad n \in N_0 \text{ (natural integer)}, \quad P_0 = P_1 = P_2 = 1.$$

The Padovan sequence plays the rule of the Fibonacci sequence related to Φ; also Ψ comes out as the limit of the ratio:

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \Psi .$$

The similar properties between Φ and Ψ become again evident considering the two spirals connected with the two numbers (Figure 7).
Van Der Laan studied 3D cubes and cuboids having measure of edges equal to 1, Ψ, Ψ^2,… till Ψ^7 and proposed this system of form as a new aesthetic canon in buildings [14, 15]. Figure 8 shows that the Padovan spiral, embedded in cubes and cuboids, is generated by triangles; all the sides of the equilateral triangles and, consequently, the Padovan spiral belong to the same plane.

[3] The name of the sequence is related to the architect Richard Padovan, who has deepened the works of Van Der Laan

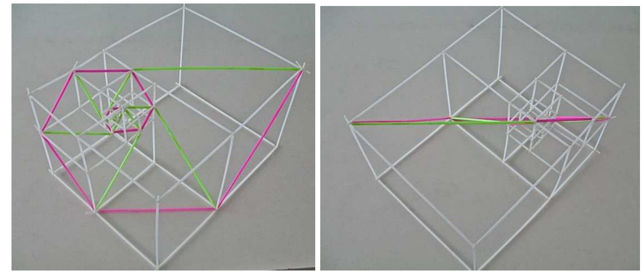


Figure 8: *Courtesy of Jim Hausman and Fabien Vienne [20]*

The pictures in Figure 9 and Figure 10 give views of two famous Abbeys designed by Van Der Laan. He involved the plastic ratios not only in the buildings but also in the furniture.

Figure 9: *Sint Benedictus Berg Abbey at Vaals (The Netherlands).*

Going back to equation (13), it's easy to verify that, if $p=2$, the golden number Φ is the positive real solution of (13).

Let us give a geometric interpretation of

(14) $a^3 - b^3 = pab^2$,

equivalent to (12): the difference in volume of two cubes with edges a and b respectively, is *p*-times the volume of a prism having height a and square base with side b ($p=1$ in Figure 6).

Another possible interpretation is obtained multiplying the (14) by $\frac{4}{3}\pi$:

$$\frac{4}{3}\pi a^3 - \frac{4}{3}\pi b^3 = p\frac{4}{3}\pi ab^2 ,$$

i.e. the difference in volume of two spheres with radius a and b respectively is equivalent to *p*-times the volume of the revolution ellipsoid having semi-axes a and b.

Figure 10: *Rosenberg Waasmunster (Belgium)*

The second case. Let us now divide a segment in two parts of length a, b $(a > b)$ satisfying the proportion:

(15) $\dfrac{a^2}{b^2} = \dfrac{a+qb}{a}$ $(q \in N)$,

equivalent to

(16) $\left(\dfrac{a}{b}\right)^3 = \left(\dfrac{a}{b}\right) + q$,

consequently $\dfrac{a}{b}$ is the only real solution of the cubic equation

(17) $x^3 - x - q = 0$.

If $q = 1$, we have again the *plastic equation*; if $q = 6$, the solution of (17) is the *copper number*.

The relation (16) is equivalent to

(18) $a^3 - ab^2 = qb^3$

which can have the following geometrical interpretation: the difference in volume of the cube with edge a and the volume of the prism having height a and square base of side b is equivalent to q-times the volume of the cube with edge b.

6. STATIC PROPORTIONS IN THE *VELA QUADRABILE FIORENTINA*

The *Vela Quadrabile Fiorentina* (VQF in the following) is a portion of a sphere surface (radius r) **described in the treatise *Formazione e Misura di tutti i cieli* by Vincenzo Viviani[4] (Figure 16), written** in 1692 with the aim of determining parts of a sphere surface with rational area, being r^2 the unit of measure.

[4] Vincenzo Viviani (1622-1703), good and well-known Mathematician, was engineer by the Granduca of Toscany and the last pupil of Galileo Galilei, like he described himself *Postremo Galilei Discipulo*.

[http://www.groups.dcs.st-and.ac.uk/~history/Mathematicians/Viviani.html]

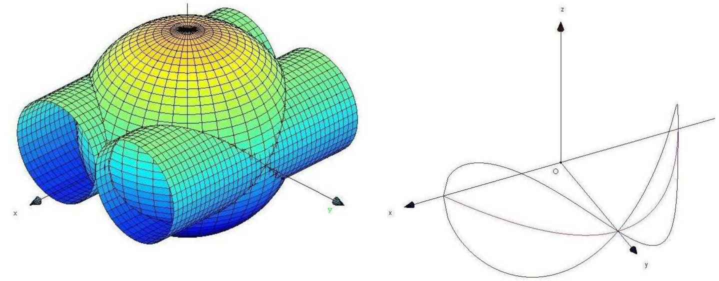


Figure 11: *Construction of VQF and the Viviani's curve*

Two VQF are obtained by cutting out from the sphere surface the four parts selected by the intersection of two cylinders of radius *r/2*, having parallel axes lying on an equatorial plane Π_0 (Oxy) (Figure 11, left).

The line belonging to the sphere and to the cylinders is composed of two Viviani's curves [11]. Consequently the boundary of the VQF is made up with two Viviani's hemi-curves.

The projection of the Viviani's curve (Figure 11, right) on the equatorial plane Π_1 (Oxz) gives the Gerono's lemniscate [10] (remember Figure 4).

We try to connect proportions with measures of geometrical parts in VQF.

The area A_v of the VQF (Figure 12, left) is equal to $4r^2$, that is the measure with respect to r^2 is expressed by a rational number.

Multiplying the relation (5) by 4 we obtain

$4a^2 - 4b^2 = 4pab$,

which means that the difference in area of two VQF, radius a and b respectively, and the area of the rectangle with sides a and b are in static proportion $4p$.

The area A_h of the VQF projection (Figure 12, right) on the equatorial plane Π_0, straightforward evaluated, is equal to $\dfrac{8}{3}r^2$ (rational measure!); consequently the ratio $\dfrac{A_v}{A_h} = \dfrac{3}{2}$ gives the *sesquialtera proportion*.

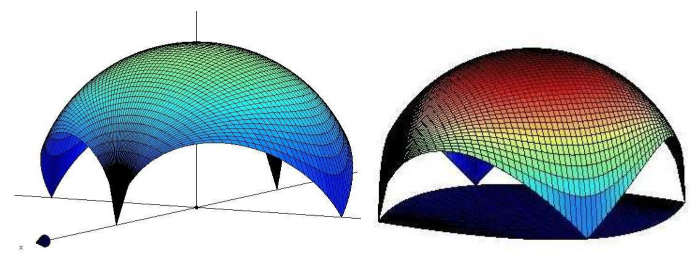


Figure 12: *Two views of VQF and its plane projection*

Taking account that the area A_g of the region bounded by the Gerono's lemniscate is equal to $\frac{4}{3}r^2$, we deduce that $\frac{A_v}{A_g}=3$ and $\frac{A_h}{A_g}=2$, both integer metallic numbers, solutions of (9) if $q=6$ and $q=2$ respectively.

Cutting away the two hemi-cylinders from the hemisphere, the volume V_1 of the solid remainder P_1 (Figure 13, left) is equal to $\frac{8}{9}r^3$ (rational measure again with respect to r^3).

Multiplying the two members of the (14) by $\frac{8}{9}$ we obtain

$$\frac{8}{9}a^3-\frac{8}{9}b^3=\frac{8}{9}pab^2$$

i.e. the difference in volume of two solid parts like P_1 (radius a and b respectively) is in *static proportion* $\frac{8}{9}p$ with the volume of the prism having height a and a square base with side b.

The solid conic part P_2 of the hemisphere, having vertex in the centre of the sphere and the VQF boundary as directrix (Figure 13, right), has volume $V_2=\frac{4}{3}r^3$, rational again. Consequently the ratio $\frac{V_2}{V_1}=\frac{3}{2}$ gives the *sesquialtera proportion*.

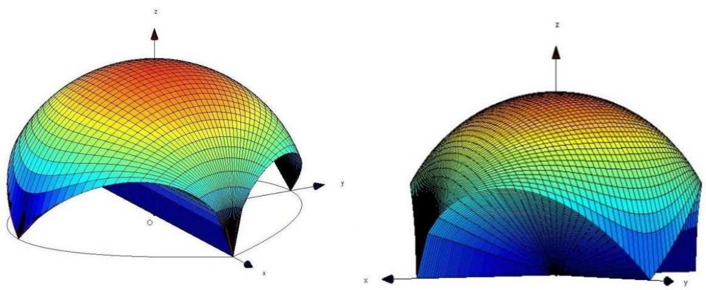


Figure 13: *Two solid parts related to the VQF*

Multiplying by $\frac{4}{3}$ the two members in (14) we obtain

$$\frac{4}{3}a^3-\frac{4}{3}b^3=\frac{4}{3}pab^2$$

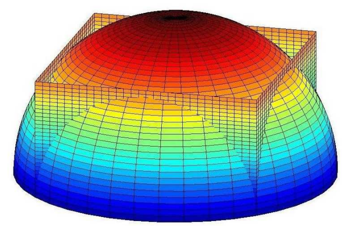
i.e. the difference in volume of two conic parts like P_2 (radius a and b respectively) is in static proportion $\frac{4}{3}p$ with the volume of the prism having height a and square base with side b.

7. DYNAMIC PROPORTIONS IN THE *SAIL VAULT*

The Sail Vault (SV in the following) is a surface obtained from an hemisphere (radius r) cutting out four half-cups, through four planes orthogonal to the equatorial circle, each containing one of the sides of the inscribed square (Figure 14). Also for the SV it is interesting to discover some proportions between measures of parts.

The area A_s of the SV is equal to $2\pi r^2(\sqrt{2}-1)$ that is its measure with respect to r^2 is given by an irrational number. Taking account that the area A_{hs} of the hemisphere is equal to $2\pi r^2$, we obtain the ratio $\frac{A_{hs}}{A_s}=\sqrt{2}+1=\theta$, that is the *silver number*. The construction of the so called *silver rectangle* is evident, starting from a square (Figure 15).

The volume V_s of the part of the solid hemisphere, covered by the SV, is equal to $\pi r^3(5\sqrt{2}-4)/6$, i.e. V_s is expressed again by an irrational number with respect to r^3. Taking account that the volume V_{hs} of the hemisphere is equal to $\frac{2}{3}\pi r^3$, the ratio $\frac{V_s}{V_{hs}}=\frac{5\theta-9}{4}$. We can suppose that the *silver number* is connected to the SV.



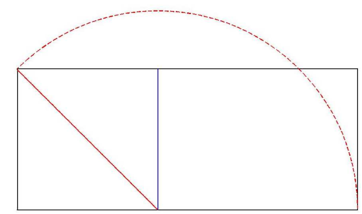


Figure 14: C*onstruction of the SV* **Figure 15:** *Construction of a silver rectangle*

Again, multiplying the relation (5) by $2\pi(\sqrt{2}-1)$ we obtain

$$2\pi(\sqrt{2}-1)a^2-2\pi(\sqrt{2}-1)b^2=2\pi(\sqrt{2}-1)pab \qquad (22)$$

that is the difference in area of two SV of radius a and b respectively is in *dynamic proportion* $\frac{2\pi p}{\theta}$ with the area of a rectangle of sides a and b.

Figure 16: *Vincenzo Viviani and his Treatise*

8. CONCLUSIONS

In this paper we propose one possible *route* to connect geometrical properties with metallic and plastic numbers, through algebraic second or third degree equations. We take occasion to introduce in the proportion theory the *Vela Quadrabile Fiorentina*, an intriguing dome already studied by Vincenzo Viviani at the end of the Seventeenth Century. We try to present in a pleasant way the proportions theory non only to specialists but also to everybody interested in mathematical aspects related to aesthetic or harmony in forms. We are sure that the contents can be appreciated at many levels: as student or teacher of basic algebra, as architects, designers, artists in planning their production.

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GUERNICA: 75TH ANNIVERSARY

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1. INTRODUCTION

In 1937, Pablo Picasso, aged fifty-six was considered world's foremost living painter. Due to his popularity the Government of the Spanish Republic commissioned Picasso an enormous mural painting of almost 8 meters width to be shown at Paris International world's Fair. Picasso's mind was blank for several months, but when Nazi air raiders destroyed the city of *Gernika*, his unchained anger led him to an intense and passionate creative process, which culminated in the most iconic of 20th century masterpieces. From the 45 sketches that served to prepare the painting and the ten pictures his partner Dora Maar took during the creation of the painting we can deduce the process followed by Picasso during the 36 days of fury in which he painted *Guernica*. Note that along this paper we will use the official Basque name *Gernika* when referring to the town and the Spanish name *Guernica* when referring to the painting, as Picasso named it in 1937.

2. THE BOMBING OF GERNIKA ON APRIL 26TH 1937

José Antonio Aguirre, President of the Autonomous Basque Government during the course of the Spanish Civil War (1936-1939) described the bombing of Gernika with these words:

> *"Gernika was a town of around seven thousands inhabitants located in a peaceful valley surrounded by mountains. It was an open town, totally lacking any defenses. Market day was held every Monday in Gernika and this was a well-known and picturesque event, where peasants and villagers congregated. This was the scenario that Franco and Germany chose to carry out the first rehearsal of a total war."*

The town was far from the line of battle and the attack was absolutely extraordinary, and unexplainable as a target. Gernika began to be a symbol for groundless savagery a few days after the 26th of April, 1937. Never before there was a better planned and less predictable tragedy at the same time.

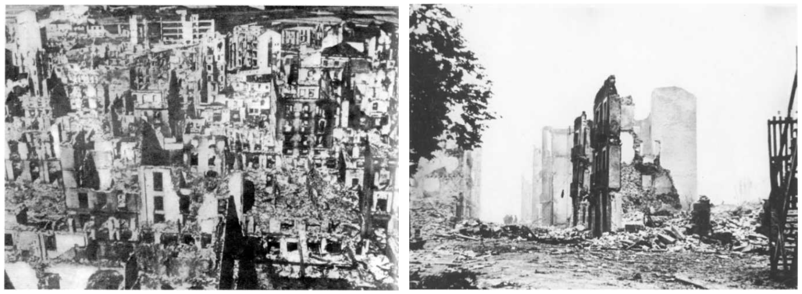


Figure 1: *The raid carried out by 42 planes of the Condor Legion from the German Luftwaffe and the Italian Fascist Aviazione Legionaria destroyed Gernika almost completely.*

George Steer, special correspondent for the times in the Basque Country describes the events:

> *"Gernika, the most ancient town of the Basques and the centre of their cultural tradition, was completely destroyed yesterday afternoon by insurgent air raiders. The bombardment of this open town far behind the lines occupied precisely three hours and a quarter, during which a powerful fleet of aeroplanes consisting of three German types, Junkers and Heinkel bombers, did not cease unloading on the town bombs weighting from 1,000 lbs. downwards and, it is calculated, more than 3,000 two-pounder aluminum incendiary projectiles. The fighters, meanwhile, plunged low from above the centre of the town to machine-gun those of the civilian population who had taken refuge in the fields".*

3. THE SPANISH PAVILION AT THE PARIS INTERNATIONAL WORLD'S FAIR IN 1937

During the Spanish civil war, the Government of the Spanish Republic decided to participate in the International Exhibition opened in Paris in May 1937. It was a strange decision when the country was in such an extreme situation that it would seem there was no time for anything apart from the war effort. Nevertheless the Government decided to participate in the event considering it as an act of propaganda. In addition its intention was to attract interest to the unstoppable rise of fascism in Europe.

The Spanish Pavilion of 1937, designed and constructed by Joseph Lluis Sert and Luis Lacasa in just four months, was a unique and unrepeatable creation, only made possible by the climate of passion, power and enthusiasm displayed by everyone involved in it.

From the first moment, the Government of the Republic wanted the collaboration of Picasso, as the most prestigious artist of the time. The first step to achieve this objective was his appointment as director of the Prado Museum, which the artist was delighted to accept. Joseph Lluis Sert then visited Picasso in Paris to commission a large mural painting that would be the center point of the building surrounded by pieces from artists like Joan Miró, Julio González or Alexander Calder. The building was to be a landmark in the history of Art and Architecture.

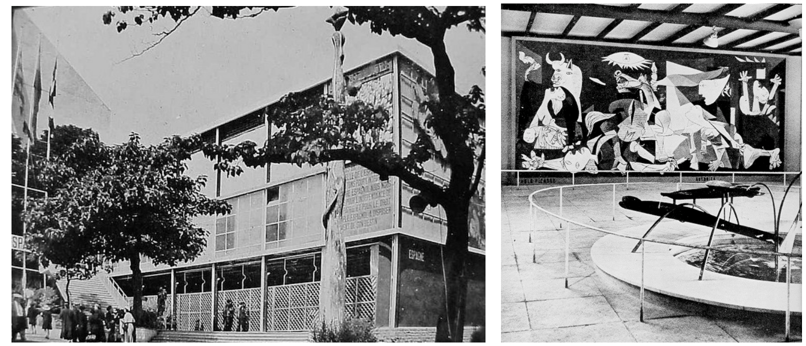


Figure 2: *The Spanish Pavilion designed by Sert and Lacasa. The Guernica Mural was hanged in a privileged position of the building, in front of Alexander Calder's Mercury Fountain.*

4. PICASSO ENTERS INTO WAR: 36 DAYS OF FURY

During the next months after the commission for the Spanish Pavilion, Picasso's mind was blank, unable to begin any work. But on April 27th all the French newspapers report the bombing of Gernika. The destruction of the Basque town is almost complete and Picasso is horrified by the tragedy. Finally, on May 1st, Picasso focuses on a single theme: Guernica. At that moment everything is clear for Picasso and it is impossible not to be involved. Now it is not a matter of politics, but a criminal act against innocent victims. That same day Picasso began the work that will have the greatest political significance of the 20th century.

Figure 3: *Composition Study for Guernica, May 9th 1937 (24x45cm).*

Picasso works with authentic fury during the first ten days, making twenty-one sketches and studies. Then, on May 11th he transfers all his fury to the canvas in his studio in Rue des Grands-Agustins in Paris. His creative impulse is burning like fire and he converts Guernica into a terrifying and sublime icon of any atrocity against innocent civilians, from Hiroshima to Bosnia, from Dresden to Vietnam. Picasso intentions with

Guernica are clear, as he says:

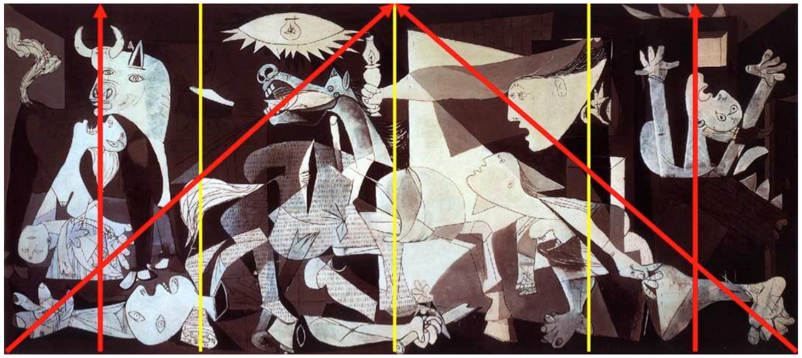
> "*No, painting is not done to decorate apartments, it is an instrument of offensive and defensive war against enemy*".

Picasso continues tirelessly painting the enormous canvas and making 24 more studies and sketches of the main characters. He will not stop until its complexion on June 6th. Some two weeks later the painting is installed in the just-completed Spanish Pavilion. During these frenetic 36 days everything that up to then was doubt is now resolved with authentic passion and anger. His biographer Pierre Daix stated: "Picasso entered into war in May 1937".

5. THE GEOMETRY OF GUERNICA

The Guernica scenery is probably inspired by *The Consequences of War*, a painting by Peter Paul Rubens. But Pablo Picasso flips the direction of the action in Ruben's painting (from left to right) by using its mirror image as starting point. Action now moves from right to the left and gets more tension and intensity due to the physiological fact that a viewer's eye naturally moves from left to right. This effect is increased with the chaos introduced by the angular distortions and the violent contrasts of light, shade, and texture.

Figure 4: *The Consequences of War. Peter Paul Rubens (up). Palazzo Pitti, Florence (1638). Pablo Picasso probably used a mirror version of this painting (down) as a starting point for the composition of Guernica.*



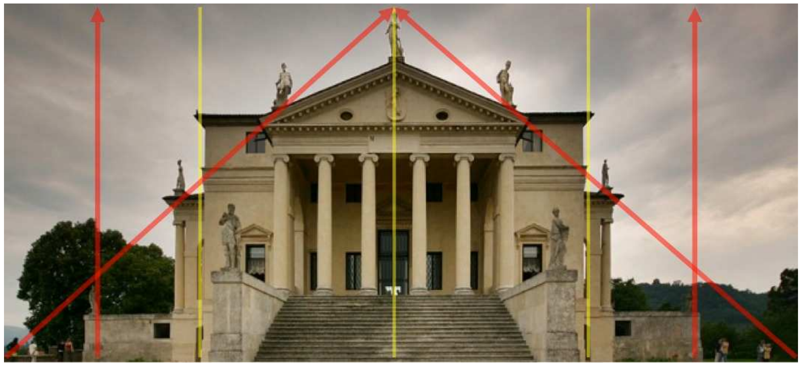


Figure 5: *The geometrical construction of Guernica can be compared with the design of many classical temples and buildings like Andrea Palladio's Villa La Rotonda (1566) inspired in the Pantheon in Rome.*

The mural is constructed on a classical scheme like the one of a Greek temple or a medieval altarpiece. Two smaller side panels frame a large central panel twice the width of the side ones. Within this framework, a large triangle dominates the composition.

The apex of the triangle is the upper central point, almost in coincidence with the rays of light departing from the small lamp. The balance achieved by the form of the running woman's knee. It is a carefully design.

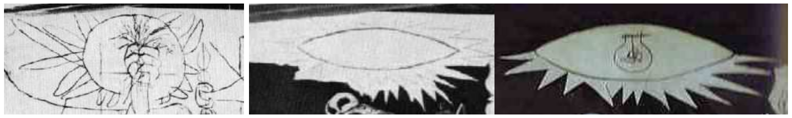


Figure 6: *Evolution of the solar disk into an electric lamp in different stages of Guernica*

It is not clear whether action occurs in the exterior or interior of a building. In the first stages of the painting a sun clearly dominates the scene and some building blocks are clearly visible. Along next stages of the picture Picasso changes the sun into an oval shape and finally he places a bulb on it. At the same time, doors and windows replace the buildings. Now it is not clear whether the scene happens inside or outside. There is no place to escape and confusion and anxiety is produced in the spectator.

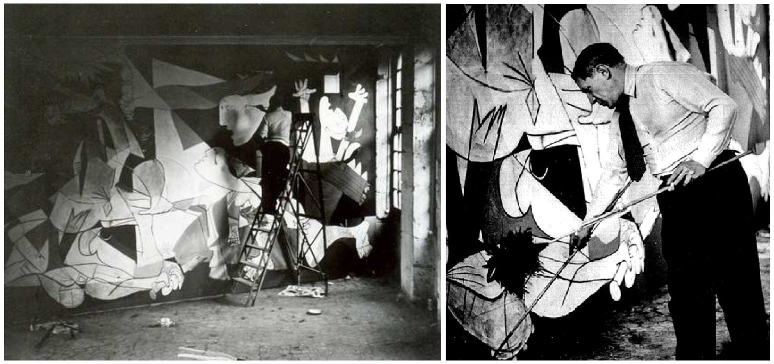


Figure 7: *Paris, May 1937. The canvas was so enormous that doubled Picasso's height. The painter needed a ladder and brushes strapped to sticks in order to paint its heights. Picasso worked in the canvas with an unusual anger and intensity, even for him.*

6. DORA MAAR'S PHOTOGRAPHS OF THE WORK IN PROGRESS

During the creation of Guernica, Picasso's companion, Dora Maar, took a series of ten photographs as the work progressed. This unique document records the magical creative process of a masterpiece. From the first moment, when the painter fits the varied composition full of figures onto the canvas, until the moment in which he finishes the painting, as we know it today, there are a multitude of transformations. He composes, decomposes, fits and disengages, until everything is in its place. In the exciting process that follows, all confusion will lead to order and harmony.

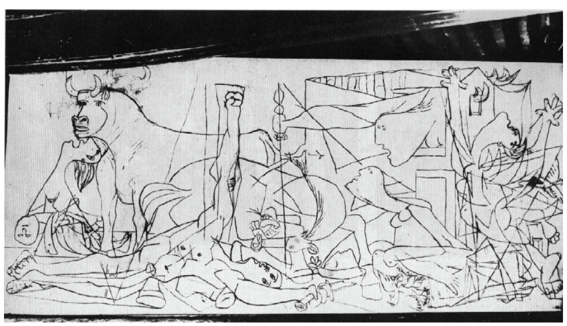


Figure 8: *Four of the ten pictures that Dora Maar took during the creation of Guernica in his Paris studio at Rue des Grans Agustins 7.*

7. THE DIFFERENT STAGES OF THE PAINTING

According to Dora Maar's pictures and Picasso's 45 sketches, the horse and the bull, which are the major elements of the composition, suffer a series of transformations both in character and placement. From a mere geometrical point of view, the bull changes its dominant position by turning its body to the left without moving its head. The flame-like tail of the bull replaces the moon placed on the left of the canvas in the early stages. The shifting of the bull's body away from the center also serves to open the space necessary to raise the head of the horse and paint a bird, probably a pigeon.

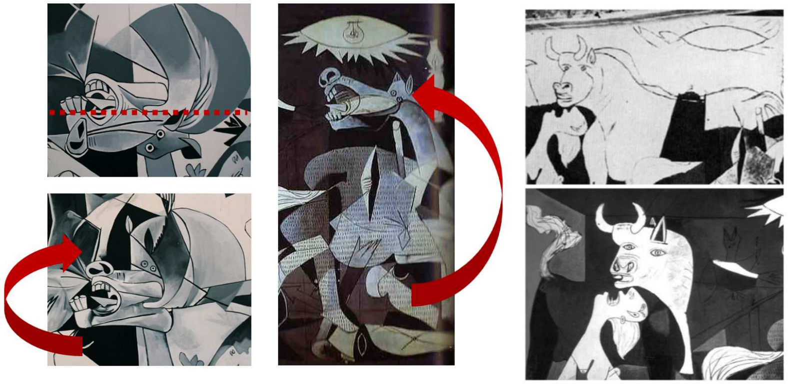


Figure 9: *The head of the horse and the head of the bull change their orientation during the different stages of the painting.*

The bull's back, insignificant as a central focus is replaced by the horse’s head, which acquires great protagonism showing its sharp tongue in a gesture of pain. The separation from the bull eliminates any chance of fighting between both animals, as in a bullfight. The bull, the only stable and unmutilated figure in the composition, the one to whom the other characters turn for help or hope, now heads straight spectator with a noble and humanized face which happens to be that of the painter.

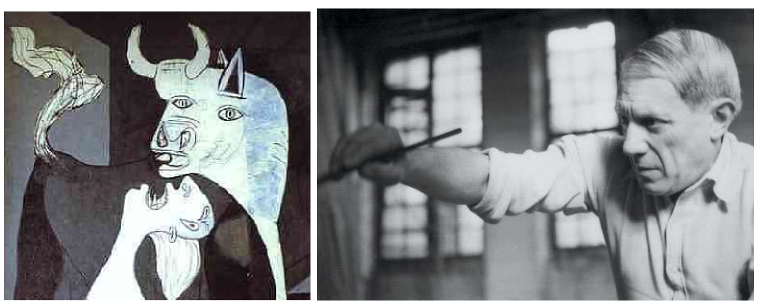


Figure 10: *Picasso appears in Guernica represented as a bull, as he did in other paintings.*

Figure 11: *The mother with the death son is sheltered under the bull’s head searching for an impossible consolation. This figure is extracted from a well-known religious scene known as La Piedad (Pietá in Italian), It depicts the supreme pain of a mother, the loss of his child. The death baby shows show similarities with one of Goya’s engraving from the series “The Disasters of War”.*

The body of the warrior is the ultimate transformation. Picasso makes on June 4th his last two sketches, a study of the hand and head of the warrior, just two days before complexion on June 6th. The dead warrior changes completely. His body is now fragmented in two pieces emphasizing total devastation. The head is ovoid, almost bare of details, which render death even more evident. A flower, rising from the broken sword can be seen as a sign of hope and rebirth. The Guernica is finally finished.

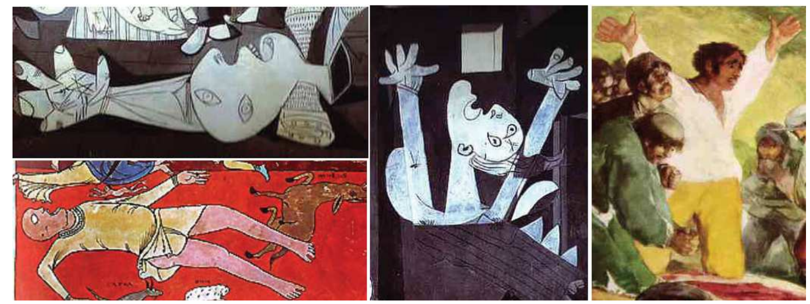


Figure 12: *Picasso got some of his ideas for Guernica from the great masters of Spanish Art. The figure of the death warrior is based on a Romanesque drawing of Saint Server Beatus (XIth Century). The woman on fire is inspired on the painting The Third of May 1808, by Francisco de Goya (1814).*

Figure 13: *Guernica. Oil on canvas. 777 cm x 349 cm. National Museum Art Center Reina Sofía, Madrid.*

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PERHAPS, AN EXTENSION

Octave Landuyt, Karel Wuytack

Division of XXX, University XXX

1. INTRODUCTION

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3.

THE EUROPEAN SOCIETY FOR MATHEMATICS AND ART ‘ESMA’ AND ‘SES RAISONS D'ÊTRE’

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1. INTRODUCTION

Objects existing in nature and not created by man all seem to possess a physical stratum. Each has a particular structure, but their functionality doesn't seem to have necessarily been studied or even recognized. Man-made objects do have functionality, indeed a multi-faceted one. Let us examine the existence of ESMA in the light of these preliminary remarks.

In order to better insure their stability through time and space, the animal species, including mankind, has developed representational tools and activities. These representations, at first interiorized and later exteriorized and materialized in more or less permanent forms, have served as our memory and sources of information. Memory of things observed exterior to us and memory of what we feel inside. These representations have a two-fold role, a double function, to be objects of memory and objects of information.

Engravings and paintings of all kinds, the oldest known of which dates back 60,000 years, are apparently the most ancient objects which have performed that double function.

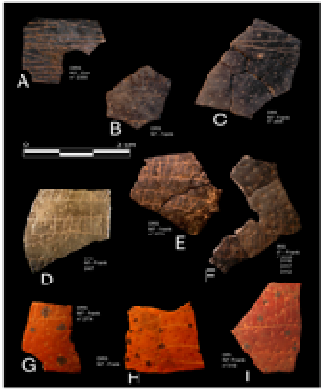

Figure 1: *Engravings on ostrich eggs (Photos Pierre Texier [3]).*

Of course, the most significant events are the first to be represented. These natural phenomena might be beneficial, such as the presence of the sun, or maleficent natural catastrophes as storms or swirling movements, related to the necessities of life or to danger and the fear of dying.

These particularly significant events are also those presenting the strongest space-time stability. They are associated with remarkable states and dynamics whose essential features have been represented. They appear with the first engravings: straight lines, curved lines, acute angles, serrated lines and undulated lines associated with repetitions and more or less periodic phenomena, like circles or spirals.

Engraving presupposes the use of a tool, probably a flint point, shaped like an arrow head and forming an exact acute angle. These first engravings exhibit elements of local geometry. Global shapes, present in cave paintings, mainly concern the animal world. More elaborate geometrical shapes, whose nature is global, such as triangles, do not appear yet.

It seems reasonable to suppose that the use of stone as the primary material for the construction of shelters or of protective constructions, such as ramparts, tombs, temples, palaces and dwellings, promoted the development of an elementary arithmetic and geometry.

The fact that the repeated use of fundamental geometrical signs was pleasing to the eye, and later, the discovery of how to fabricate clay plates and pottery, were steps leading to the development of decoration.

Here are, for instance, two pieces of pottery, symbols of femininity, created more than 2000 years before the Common Era (B.C.):

Figure 2: *Two pictures taken at the National Museum for Archaeology in Athens.*

Were these the result of successive attempts of the artist, of artists at the same worksite or of artists at worksites separated in space and time? Did the works necessitate preliminary plans and calculations? To what extent does the sexual triangle play a role in the conceptualization of the triangle as a mathematical object?

By briefly evoking the past, and by asking these questions which concern the relations between mathematics and art, where natural philosophy, archaeology, the history of art and the history of mathematics are involved, I wanted to call attention to one of the essential functions of ESMA, its cultural function.

It is directed in several directions. But before describing them, I would like to say a word about the conditions which allow ESMA to fulfill its function, in short, to make an inventory of fixtures and at the same time to give the potential elements of its physical stratum.

2. ESMA'S STRATUM

ESMA is in no way a profit-making organization. Its statute is that of an association, i.e. a group of people who share similar judgments, convictions and wishes, and who agree to give some of their time toward the realization of common objectives.

ESMA will become stronger, more active and more influential as it benefits from the help of a greater number of adherents who are competent, devoted and understanding.

Considering the material, organizational and psychological state of our societies, their evolution, present difficulties and those to come, and considering the original nature of ESMA members' activities and their multiple competencies, we know that ESMA membership will remain numerically small.

Although ESMA's register contains nearly one hundred people, as of today only about forty are active, paying members (exactly 38 in 2011). Could this numerical weakness be balanced by financial solidity?

This is not the case now for several reasons. Dues (30 euros per member) are intentionally weak. Our activities do not always correspond to projects supported by public funds. The decision-makers, still suspicious of mathematics, do not understand the cultural and social interest of these activities. The requests for support are too restricted, limited to the French part. And finally, we have no contacts within the financial circles.

This situation could change, depending on an evolution of ESMA's reputation, an understanding of its activities, the quality of our propositions and how we publicize them.

Taking into account our ethics, our training and our occupations, the hope of progress depends almost entirely on the reputation obtained through our website, its presentation and its content. These should be commensurate with ESMA's ambitions and hopes. Everyone's help is needed to maintain this reputation.

We hope that with time, ESMA will receive more attention from various public services as well as from private foundations, and even from sponsors. The UNESCO patronage will undoubtedly aid in strengthening ESMA.

After two years of existence our balance sheet is rather encouraging. Neither the objectives exposed by ESMA, nor the content of its website has received any criticism to my knowledge. Hopefully, they have contributed to giving a good international image of ESMA. I would like, apropos, to pay tribute to our first webmaster. The curves of requests to receive the Newsletter and the visits to our website have been on a constant rise.

ESMA's existence has revived interest in the important ARPAM project [1], of which ESMA is a principal partner, giving a new dimension to it. This is being accomplished in collaboration with the Steklov Institute, the Independent University of Moscow, and the Academy of Architecture of Moscow to visualize and to materialize the project in Kaluga. In addition, the Hellenic Mathematical Society wishes to set up the project. More details about these projects can be found on the ARPAM and ESMA websites. May I add that this project not only works as a kind of museum for mathematics, but by its content has a much larger and deeper artistic and cultural interest reaching all layers of society.

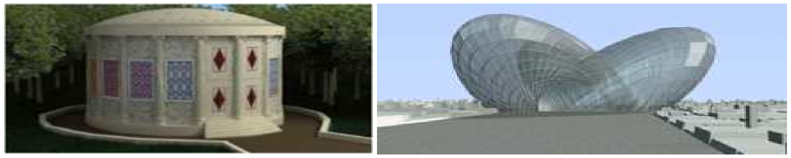

Figure 3: *The Seventh Temple and 'Brioche de Boy'.*

From its history and by its very nature, ESMA has inherited an important collection of plastic works from ARPAM. These were created as much by mathematicians endowed with artistic talent as by artists remaining true to an ancestral tradition and attracted by the pure beauty of mathematical shapes.

The image of this collection of works deserved to be improved. This has been accomplished first in the form of exhibitions, which were much appreciated by the visitors. The comments in the visitors' book at the last large exhibition in the Town Hall of the fifth administrative district of Paris give evidence to this. Allow me to present a few of them:

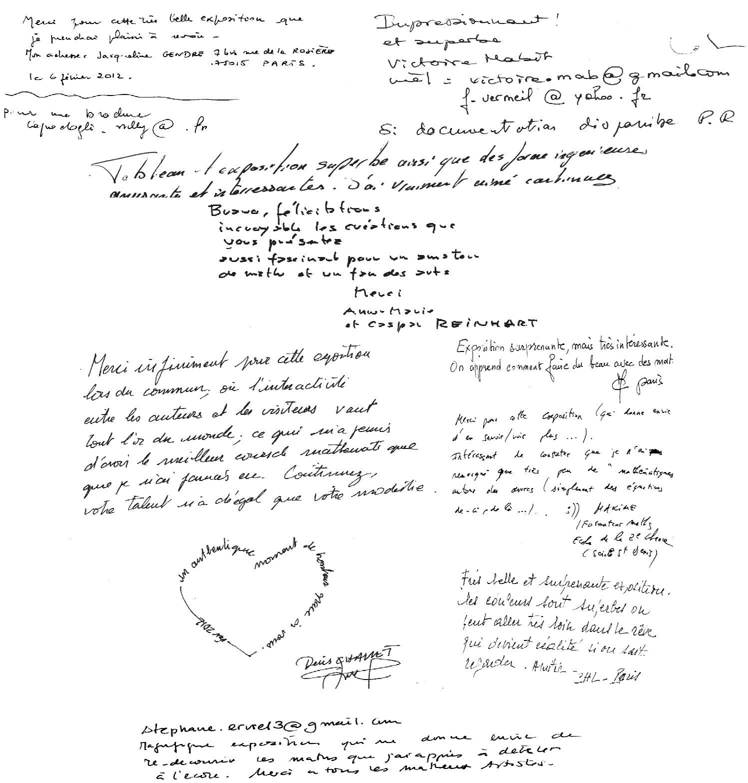

Figure 4.

Reading some of these comments, one can appreciate the interest of these exhibitions for the public at large, reconciled with mathematics. From its new look at mathematics, the influence of this public of adults on the young generations can only be beneficial.

To reach the former, whose formation of understanding and taste is still developing, oral presentations, glowing and animated, are necessary. By making modern mathematical notions accessible to the audiences, by showing them both beautiful and astonishing objects, these exposés, also based on elements of the history of art or of mathematics, win the support of children of all ages without reservation.

The article published in the 'Gazette des Mathematiciens' of October 2011 [2], and our newsletters refer to this success through the enquiries that were made and through more recent ones. If more convincing were needed, we could quote these two testimonials:

> «all the students, teachers and the principal of Nikaia's 9th high school would like to deeply thank you for your presence here. We really enjoyed your speech, and acquired many teaching experiences by your educational methods. We would be grateful to hear from you as your knowledge makes as wiser.
> Sincerely yours,
> Katerina Glinou, principal of Nikaia's 9th junior high school»

The ARPAM project, these exhibitions and exposes without a doubt characterize the originality of ESMA compared to other similar institutions.

In a recent interview in the newspaper «Le Monde», Jean Clair, author of «Hubris. La fabrique du monstre dans l'art moderne. Humoncules, géants et acéphales» says that «What is interesting in modern and contemporary art is that monsters, the new Titans, seem to have triumphed over gods, and that ugliness seem to have taken over beauty as the element of abstraction and fascination».

People familiar with math-art works might disagree with this point of view: as we have just seen, these works, far from being monsters, are commonly qualified as beautiful by the very few who have seen them. ESMA exhibitions also help to reconcile the public with beauty. That is a supplementary argument in favor of ESMA's existence and publicity for its exhibitions.

3. THE FUNCTIONS OF ESMA

This glimpse on the recent ESMA activities emphasizes one of the most important reasons for which ESMA was created: to contribute to lowering the psychological barriers which separate mathematics from the public, whatever it may be, by drawing on the charm that the beauty of a successful materialization of mathematical objects can exert on everybody.

Mathematicians who practice art and artists who do mathematics for intellectual and visual pleasure, find here an image-enhancing social justification for their creative work. Of course, the content of the exhibitions and the mathematical material accompanying them will be enhanced as the works become more numerous and varied.

It is thus justified to create an entity liable to encourage and to facilitate these creations by gathering all the participants, by striving to give them larger visibility, and when possible, by helping them not, including technical, material and financial assistance if necessary.

In this setting, contributing to the training of new participants through adapted schools is a part of the ESMA objectives.

Two other institutions display programs similar to those of ESMA. These are ISAMA and especially, BRIDGES, beautiful names indeed. They apparently came into existence after ARPAM, the parent of ESMA. It seems also that the recent evolution of BRIDGES in particular was influenced by the presence and the characteristics of ESMA.

Moreover, these institutions are American. In the past of years of plenty, some Europeans could go to the States and the Americans had no difficulty coming to Europe for academic tourism and exchange. These exchanges have become more difficult for financial reasons.

Therefore, it seemed right to give particularly to those encountering financial difficulties, new opportunities to meet and to exchange, thus bringing them a form of understanding and psychological support to their work.

In addition, there was no reason to assume that only American institutions should have the monopoly of these activities. Europe has inherited a long artistic and productive tradition close to mathematics from the Greeks, the Renaissance, and the more recent cubist period. Europe has its own resources and its own creativity, which deserve to be more organized and supported.

We were thus in the presence of very many reasons for justifying the creation of a European institution, an institution that would have wider influence by going beyond the more limiting local activities.

And finally, it fully fits the vision, notably carried by science, mathematics and by art, of a more united world, whose internal dissensions echo all the forms of lack of intelligence and of narrow-mindedness in space and in time. Creating the European Society for Mathematics and the Arts also underlined the wish to witness the construction of a more federal architecture, a better built Europe, more efficient, outstanding on the human and cultural level.

4. ACKNOWLEDGMENT

Another thank to Sharon who translated my Franglais into English.

5. REFERENCES

[1] Bruter C.P., *Athens Conference (The Arpam project)*, http://www.math-art.eu/documents/pdfs/THE%20ARPAM%20PROJECT.pdf .
[2] Bruter C.P., *Les beaux-arts au service des mathématiques*, Gazette des Mathématiciens, N° 130, Octobre 2011, 83-90 (http://www.math-art.eu/documents/pdfs/Gazette2011.pdf).
[3] Texier P. et alii, *A Howiesons Poort tradition of engraving ostrich eggshell containers dated to 60,000 years ago at Diepkloof Rock Shelter*, South Africa, PNAS, April 6, 2010 | vol. 107 | no. 14, 6180–6185.

MATHEMATICAL MODELS PAPER

Konrad Polthier

Division of XXX, University XXX

1. INTRODUCTION

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2.

3.

POSSIBILITIES OF THE SURFER PROGRAMME IN MATH ART, EDUCATION AND SCIENCE COMMUNICATION

Anna Hartkopf

Division of XXX, University XXX

1. INTRODUCTION

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MATHEMATICS AND ART WITH MITER JOINTS AND 3D TURTLE GEOMETRY

Tom Verhoeff

Division of XXX, University XXX

1. INTRODUCTION

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2.

3.

FINDINGS ABOUT LEONARDO DA VINCI

Rinus Roelofs

Division of XXX, University XXX

1. INTRODUCTION

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2.

3.

TRANSFORMING POLYHEDRA

Xavier De Clippeleir

Designer, Design Academy Eindhoven.
xdc2000@hotmail.com

1. INTRODUCTION

Slinky, the metal spring toy that walks down stairs has been my key inspiration. When squeezed it forms a cylinder with an ellipse as cross-section (figure 1). The endings remain circular, parallel or anti-parallel. The truncated parts are assembled to make flexible chains. They led to the discovery of several transforming polyhedra and polyhedral structures.

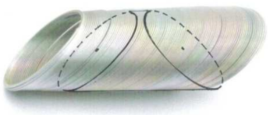


Figure 1.

2. RING OF ELLIPTIC CYLINDERS

Figure 2 shows a closed ring of 12 anti-parallel Elliptic Cylinders. The parts rotate into different forms: a square, a circle, a triangle, an octahedron and countless others. The ring is produced by the Swiss toy company NAEF since 1983, named ELLIPSO.

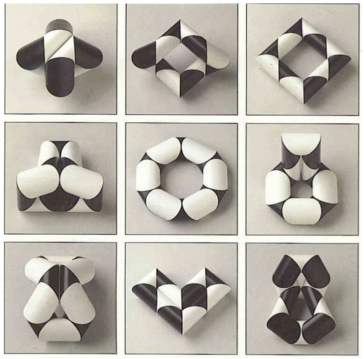


Figure 2: *ELLIPSO, beach wood.*

3. CUBE

The Elliptic Cylinder was the subject of my degree work at the Royal College of Art in 1978. After experimenting with many flexible structures I decided to build a cube with elliptic cylinders as edges. The edges were cut under the suitable angle of 35° 16' to get two anti-parallel circular sections. The parts were provided with axes of rotation, 24 in total. This geometry allows the cube to transform into a new polyhedron in which we recognize 12 faces, 24 edges and 14 vertices. The transformation is rigid. Turning one of the axes makes the whole cube follow. The black corner parts remain parallel during the transformation. Each axe of rotation is parallel to a diagonal of the cube. This observation turned out to be the recipe for the discovery of the transformation of the other rhombic polyhedra.

Figure 3: *Three positions of the transforming cube (35 cm, beach wood).*

4. CUBICAL STRUCTURE

The prototype of the cube from 1978 was reproduced in a small series by the Swiss toy company NAEF in 1992. The vertices of six cubes were provided with rotation axes and assembled. The structure transforms similar to a single cube. Each cube has 32 rotation axes: 2 per edge plus 8 corners. The direction of rotation of a single cube in the structure can be chosen: to the right or to the left. The corner parts behave as if glued together or turning in opposite direction. This results in different symmetries.

Figure 4: *Three positions of 6 assembled cubes*

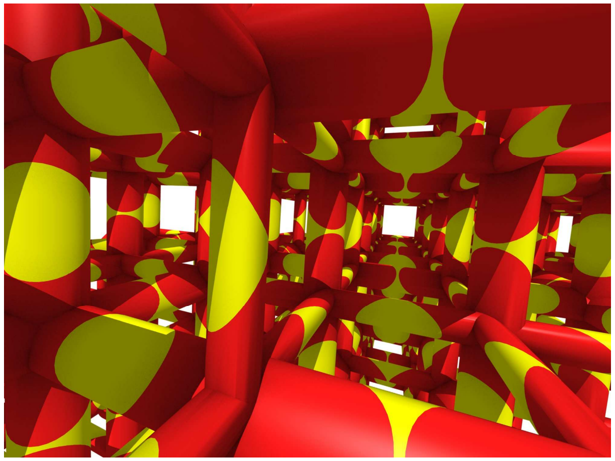


Figure 5: *View inside the transforming cubical grid. CAD image by W.Scheurman.*

5. RHOMBIC DODECAHEDRON

The 24 edges of the dodecahedron are provided with 2 folding lines. The polyhedron transforms into a cube. The rhombic dodecahedron is a space filling solid. Its lattice transforms in analogy to the cubical.

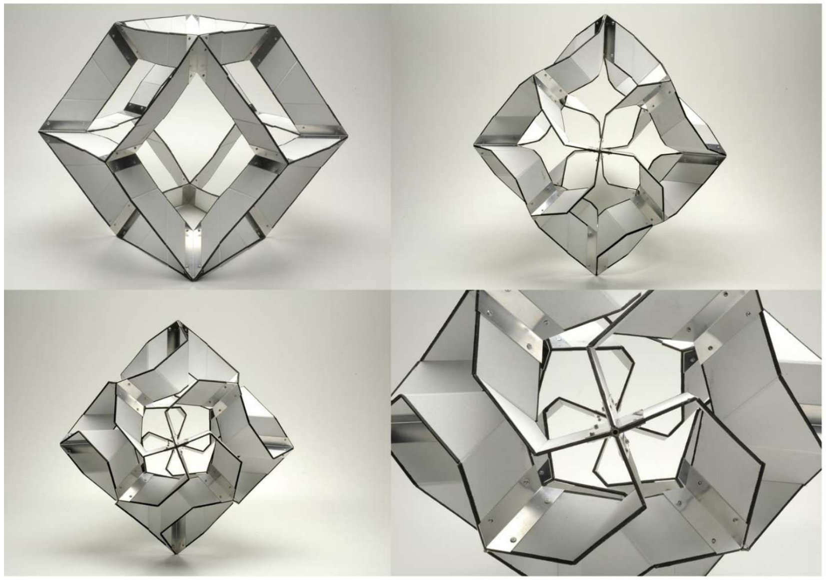


Figure 6: *Open and closed positions of the rhombic dodecahedron. Bottom left: the cube position. Model: 60 cm, canvas, cardboard, aluminium.*

6. TRIACONTAHEDRON

The rhombic triacontahedron has 30 faces. The 60 edges are provided with two folding lines.The triacontaheron rotates into a dodecahedron (Figure 6).

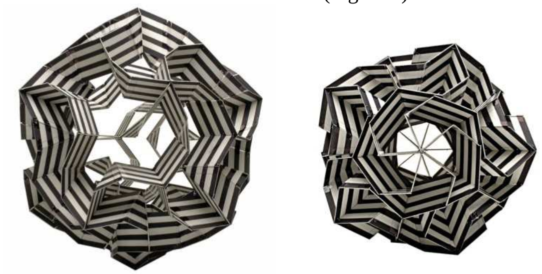


Figure 7: *Medium and closed positions. Model: 90 cm, striped cardboard.*

NOTE THE RHOMBIC HEXECONTAHEDRON (60 faces, 120 edges, 240 hinges) is also a candidate for transformation. A balsa wood model is being made. I presume it makes the total set of edge divided transformable polyhedra complete.

7. SPHERES

The concept to transform the cube is applied to the sphere. The sphere is divided in 8 parts and connected with 24 axes. The structure open and closes. The model has been printed with integrated hinges, out of one piece.

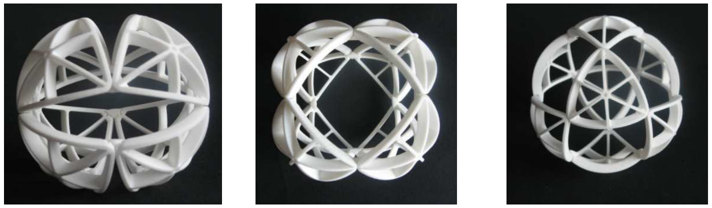


Figure 8: *Transformed sphere.*

8. ACKNOLEDGMENT

I would like to thank Jean-Jacques Stiefenhofer (designer dipl. HfG Ulm) for introducing me to the world of geometry and its importance for design.

Figure 9.

9. REFERENCES

Bruijn, N.G., *Mathematics around the designs of X. De Clippeleir*, Eindhoven University of Technology Department of Mathematics, May 2006.
Cundy, H.M., and Rollett, A.P, *Mathematical Models*, Oxford at the Clarendon Press, 1951.
Hilbert, D. and Cohn-Vosse, S., *Geometry and the imagination*, Chelsea New York, 1952.

MATH VISUALISATION: BRIDGING THE GAP

Jos Leys

Mathematical Imagery
www.josleys.com

1. INTRODUCTION
2.

Computer graphics enthusiasts have the skills to produce images and animations, but often lack advanced mathematical knowledge. For mathematicians, the opposite is true.
In the field of mathematical visualization there typically is little interaction between creators of imagery.

In this presentation I give three cases from my own experience that show that the collaboration between professional mathematicians and computer graphics specialists, and also the collaboration of computer graphics specialists amongst themselves, can lead to interesting results.

Case 1 is about my collaboration with David Wright, one of the authors of '*Indra's Pearls*', a book on the limits sets of groups of Moebius transformations. We worked on extending the code presented in the book, and on the related subject of spirals of tangent circles in the plane.
Case 2 describes some of the results of online collaboration through the Fractalforums website.

Case 3 is about my work with Etienne Ghys of the ENS-Lyon: graphics for his public lectures, and the 'Dimensions' film. There is now a new film in the making due to be released in September 2012.

2. DETAILS

The presentation follows on the subsequent pages and is self-explanatory.

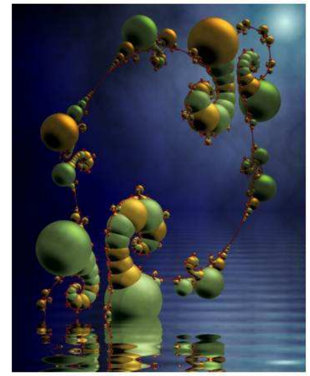
Math visualization : bridging the gap

Jos Leys

Math Art Summit, Brussels
May 2012

Where is the gap?

- Between computer graphics enthusiasts and mathematicians.
- Between computer graphics enthusiasts themselves.
- The bridge : Intense collaboration can lead to interesting results.



Who am I ?

- Retired engineer
- Hobby: math visualization
- www.josleys.com

Mathematical Imagery by Jos Leys

Home | Galleries | Articles | References | Forum | Links | About

Latest Galleries :
- 3D Newton fractals
- Mandelbrot tribute 2 ...through another 57 images in color
- Tribute to Benoît Mandelbrot ...through 51 images
- Kaleidoscopic IFS 32 amazing fractal 3D objects
- Mandelbox A special kind of 3D fractal

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Newton type fractals in 3D
Posted on 2011-01-17 18:17
Newton's method for finding the roots of an equation creates fractals if the variable in the equation is a complex number. I found a way to show these fractals in 3D. See this new gallery.

Another tribute to Benoît Mandelbrot
Posted on 2010-11-09 12:55
I further improved my algorithm for 3d views of Mandelbrot and Julia sets. See this gallery.

A tribute to Benoît Mandelbrot.
Posted on 2010-10-24 16:52
Benoît Mandelbrot passed away on October 14, 2010.
He will be remembered in the first place for his contributions to science, but he also gave us the magnificent beauty of the Mandelbrot set. As a tribute, I have compiled a collection of images of the set in a 3D form. See this gallery.

Kaleidoscopic 3D IFS fractals
Posted on 2010-09-07 23:52
As a follow-up to the article, ("A fractal soccer ball"), about "kaleidoscopic 3D Iterated Function System fractals", I have now opened a new gallery with 32 images.

A fractal soccer ball
Posted on 2010-06-21 12:45
A new article, "A fractal soccer ball", about "kaleidoscopic 3D IFS fractals", was published on the site of the CNRS.

***Dimensions* wins the Prix d'Alembert.**
Posted on 2010-06-21 12:30
The film "Dimensions", that I made together with Etienne Ghys and Aurélien Alvarez, won the "Prix d'Alembert" a bi-annual price given by the French Mathematical Society to the best projects in math vulgarization.

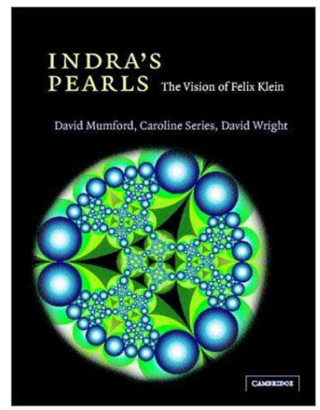
The Mandelbox
Posted on 2010-05-28 11:53
I wrote another article (again only in French, sorry!) about this special kind of 3D fractal.
There now is a small new gallery with some images on this subject..

Case 1 : Indra's Pearls

- Published 2002, with accessible math!
- Limit sets of groups of Moebius transformations.

$$f(z) = \frac{az+b}{cz+d}$$

- The book explains a generic code for the images.

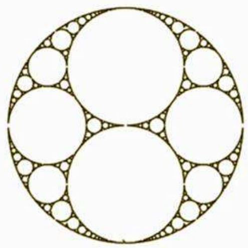


- Example

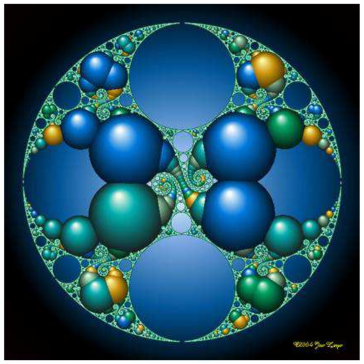
$$f(z) = \frac{z}{-2iz+1}$$

and

$$g(z) = \frac{(1-i)z+1}{z+(1+i)}$$

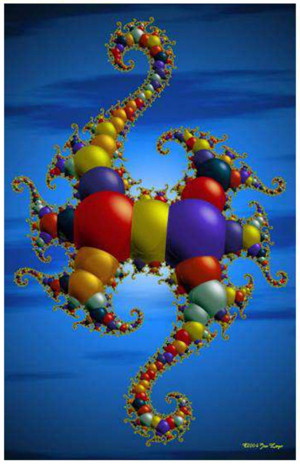


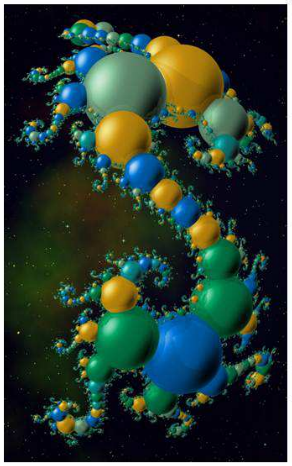
…spheres instead of circles…

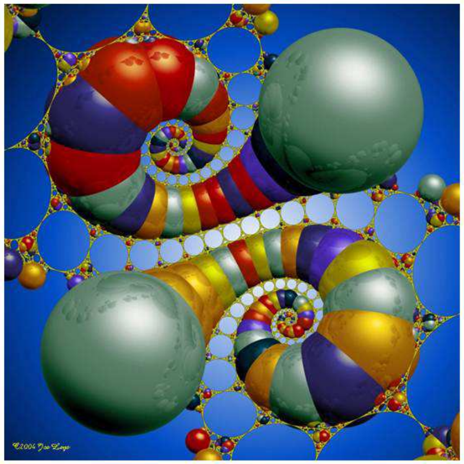


Extending the code

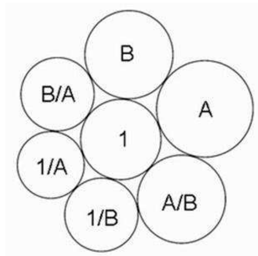
- Correspondence with David Wright at Oklahoma State U.
- A long string of e-mails to produce pictures not found in the book.

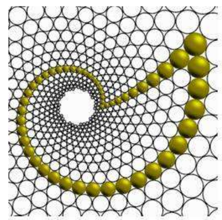




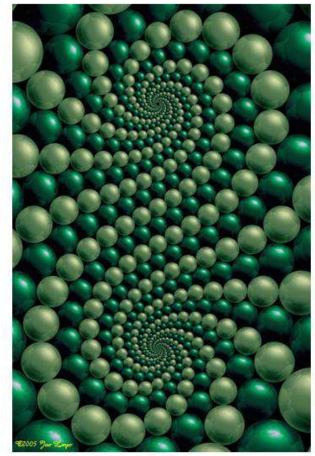


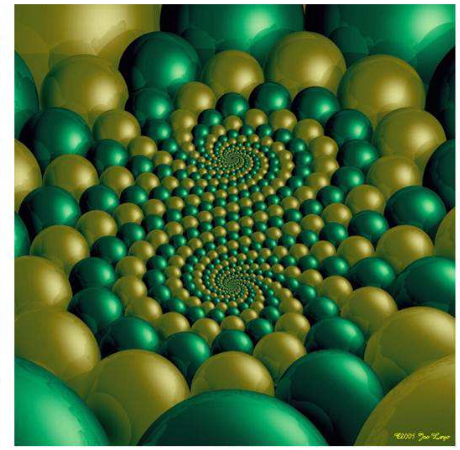
- Doyle spirals
  - A circle of radius 1 can be surrounded by 6 tangent circles with radii as in the figure.
  - If $B^q = A^p$, (q and p integers) this will produce an hexagonal tiling of tangent circles.



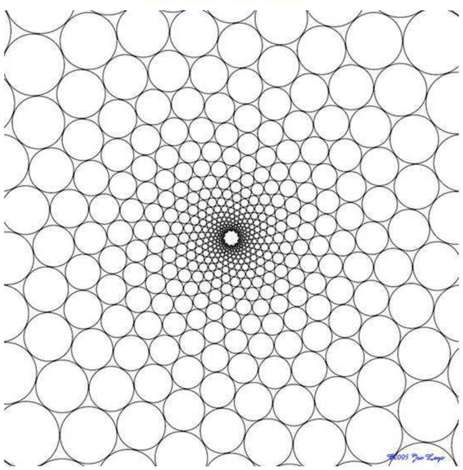


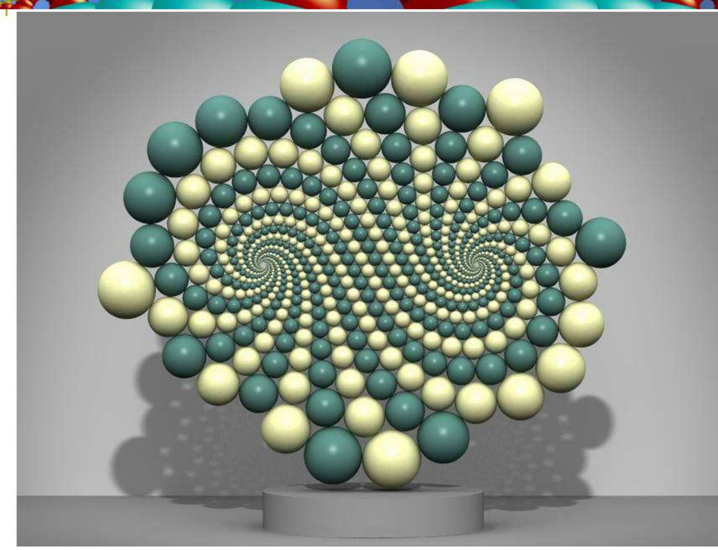
..replace circles by spheres and add an inversion



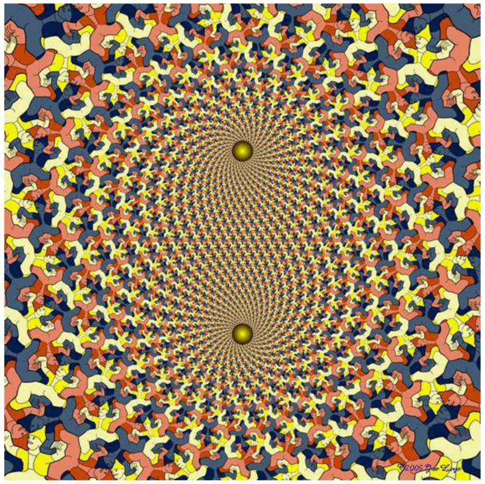


…like this…





..or use it to produce an Escher-ish tiling

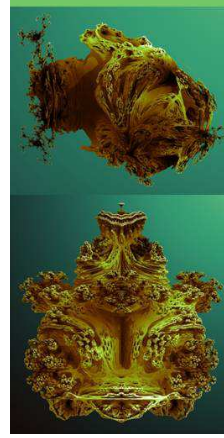


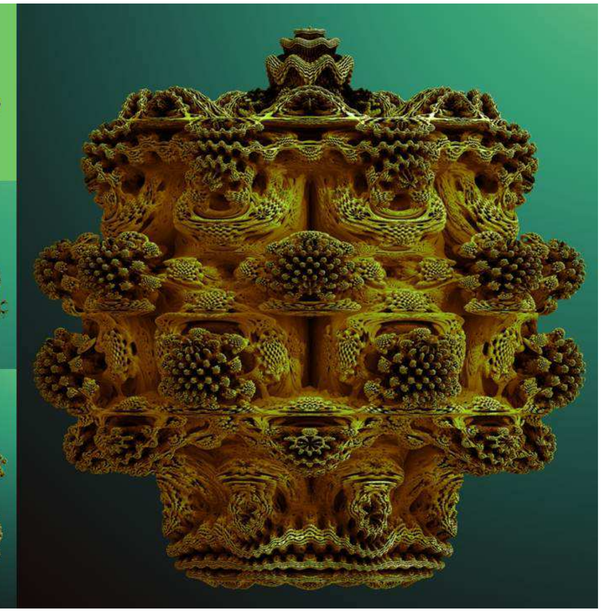
Case 2: Fractalforums

- An internet forum where an amazing amount of novel objects have been invented.

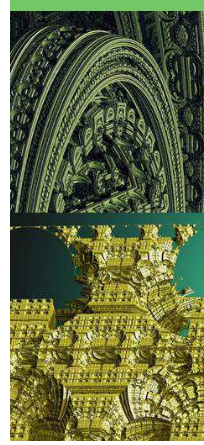
Ideas are launched and are improved upon and extended, often leading to new ‘discoveries’.

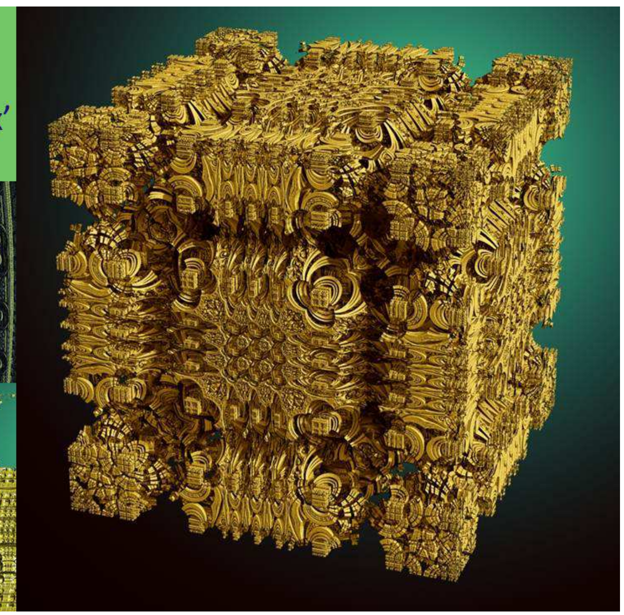
The ‘Mandelbulb’

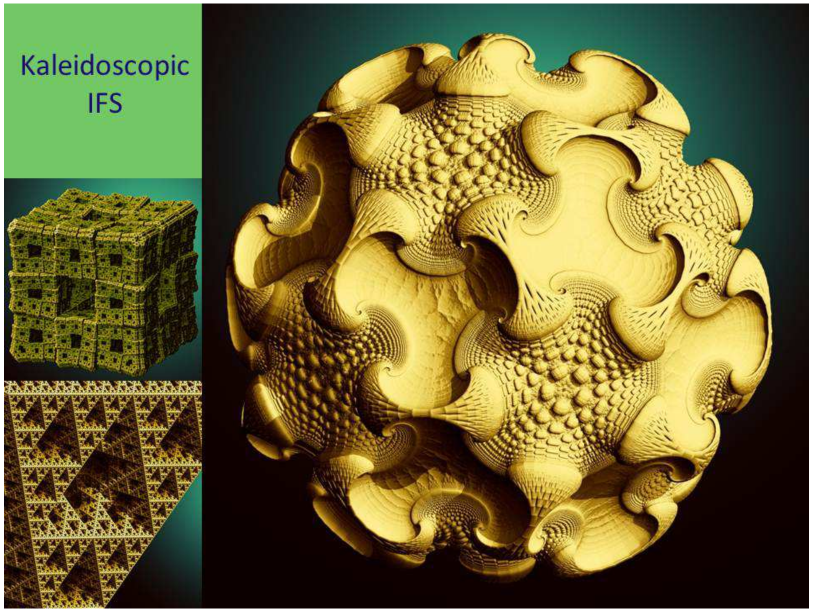




The ‘Mandelbox’



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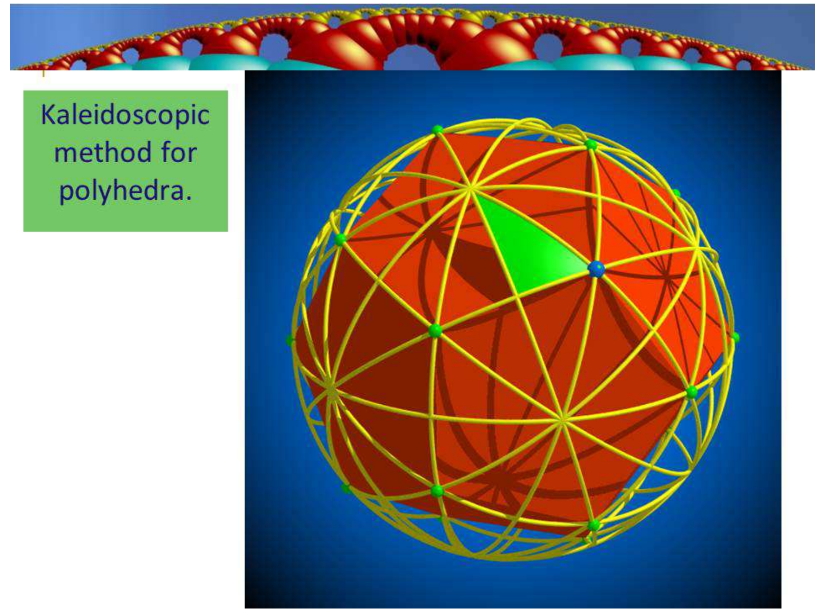
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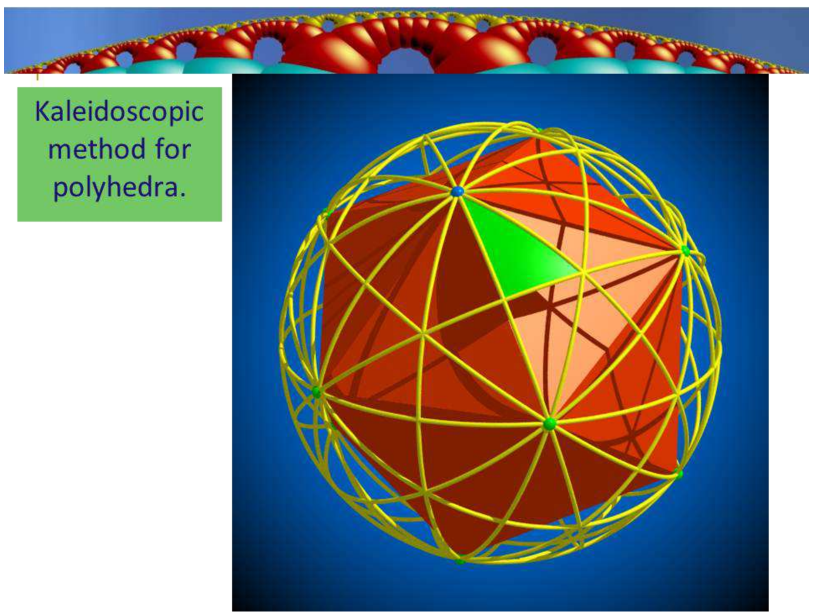
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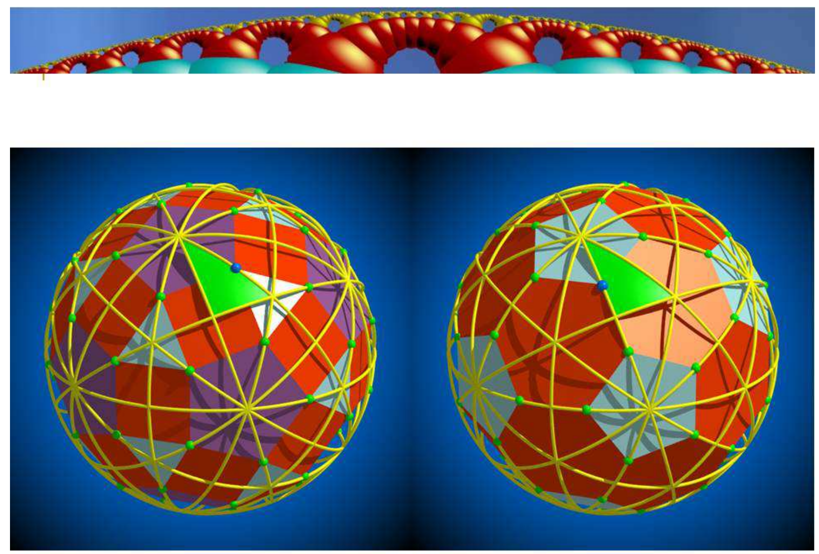

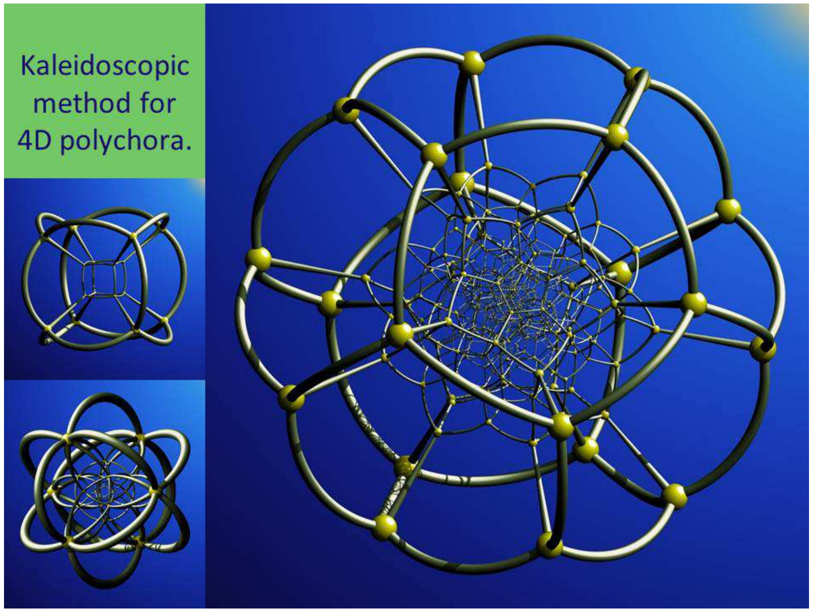
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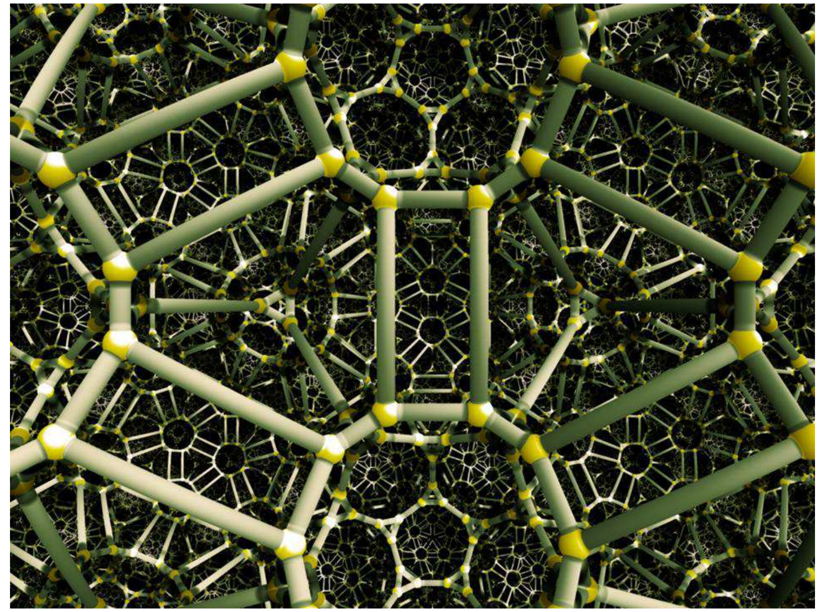
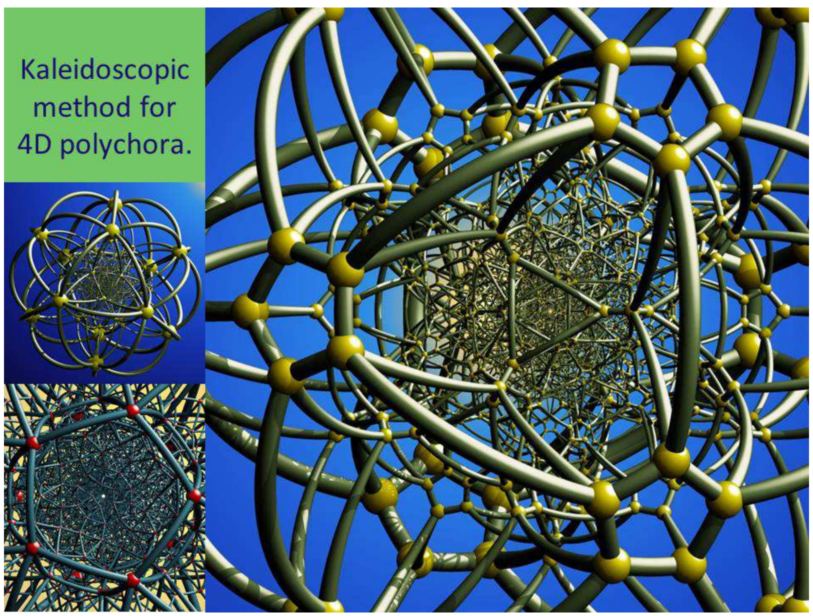
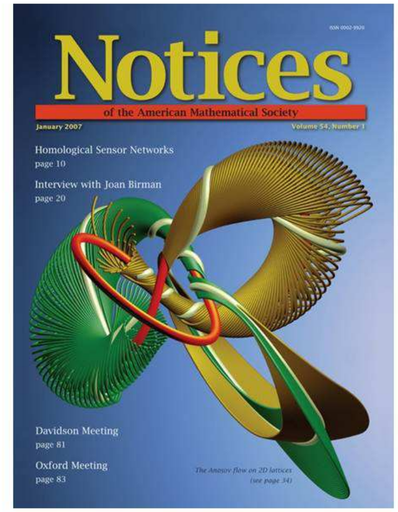



Kaleidoscopic
method for
4D polychora.

Case 3: Étienne Ghys

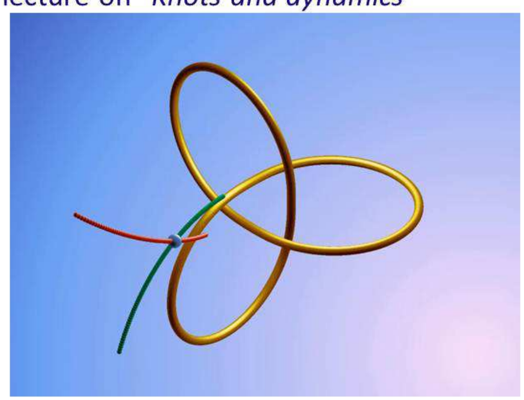
- *Directeur de recherche* at the ENS, Lyon
- Interests : dynamical systems, geometry.
- Math visualization and divulgation.



First collaboration

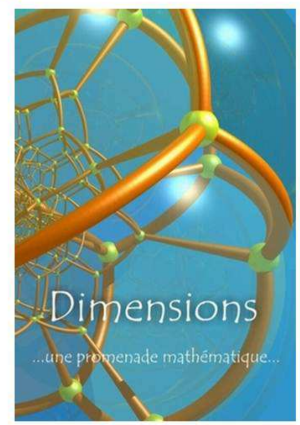
- International congress of mathematicians, Madrid 2006.
- Ghys' plenary lecture on "*Knots and dynamics*"

INTERNATIONAL CONGRESS OF MATHEMATICIANS MADRID 2006



"*Dimensions*"

- 2008 film project : 2 hours of animation.
- Non-profit : Free download
- Over 10,000 DVD's distributed
- Website : 1.3M visits in 4 years
- Prix d'Alembert 2010



MASACCIO'S TRINITY FRESCO: THE BLUEPRINT OF BRUNELLESCHIAN PERSPECTIVE

Patrick Seurinck

Division of XXX, University XXX

1. INTRODUCTION

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BULKY LINKS GENERATED BY GENERALIZED MÖBIUS LISTING BODIES

Johan Gielis

Division of XXX, University XXX

1. INTRODUCTION

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THE EYE OF THE PAINTER: DECIPHERING ART MATHEMATICALLY

Ingrid Daubechies

Duke University, Durham, USA.
ingrid@math.duke.edu

1. INTRODUCTION

This paper is the English translation of a paper that appeared in May 2012 in the Dutch magazine EOS (the images shown here are traced copyright free images of the original quality color reproductions in EOS). Through this paper, author Dirk Huylebrouck wanted to draw attention to the then forthcoming talk at the Royal Flemish Academy of Belgium. Wavelets may now be well-known, since they were applied in the JPEG-2000 format. They are used by the FBI for the management of its immense fingerprint files and by the digital cinema in Europe and the U.S. Yet, the research about the digitalization of painted masterpieces by Vincent Van Gogh, Goossen van der Weyden or the Van Eyck brothers is still emerging, and this was the topic of the talk given at the Academy.

2. WAVELETS

In 1822 the Frenchman Joseph Fourier wrote the essay 'Theory analytique de la Chaleur' ('Analytical Theory of Heat'), which literally and figuratively caused waves. Fourier claimed almost every phenomenon can be described as a combination of waves (albeit sometimes infinitely many waves are needed). His theory was implied, so to say, that 'everything vibrates'.

Fourier analysis became a very widespread mathematical theory, but over time some drawbacks became noticed. As a matter of fact, functions with a sharp decline often need the combination of a lot of waves before a good approximation can be obtained. In the example given in the illustration, a good approximation was obtained for the horizontal parts, but at the beginning and the end of each horizontal line segment the approach nevertheless always remained very considerable.

The theory provides a solution to go around this phenomenon. Instead of an infinite number of eternally up and down undulating waves, the wavelet analysis uses shifts and stretches of a well-chosen 'mother wavelet' over a finite and bounded distance. Thus, no infinitely extending undulating waves combined, but wave packages, 'tadpoles', so to say: little waves, or wave-lets. This simplified image processing, following a procedure having more and more applications, such as in painted art.

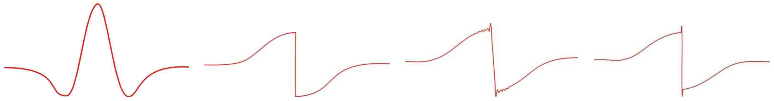

Figure 1: *1. A typical wavelet, called 'Mexican hat' (well yes!). 2. A function with a steep decline. 3. The same function approximated by 65 Fourier coefficients, or by 18 wavelet numbers, and yet the latter better.*

3. CHALLENGE

Computer processing of digital art images is a rapidly expanding interdisciplinary field. In 2007 the IP4AI workshop was organized in Amsterdam, co-organized by the Van Gogh Museum. 'IP4AI' stands for 'Image Processing for Art Investigation'. Part of the meeting was a challenge issued by the Dutch TV program NOVA: who could distinguish a Van Gogh copy from some authentic works? The copy was made by the Dutch specialist in paintings restoration Charlotte Caspers (born in Ghent, Belgium in 1979; site: www.charlottecaspers.nl).

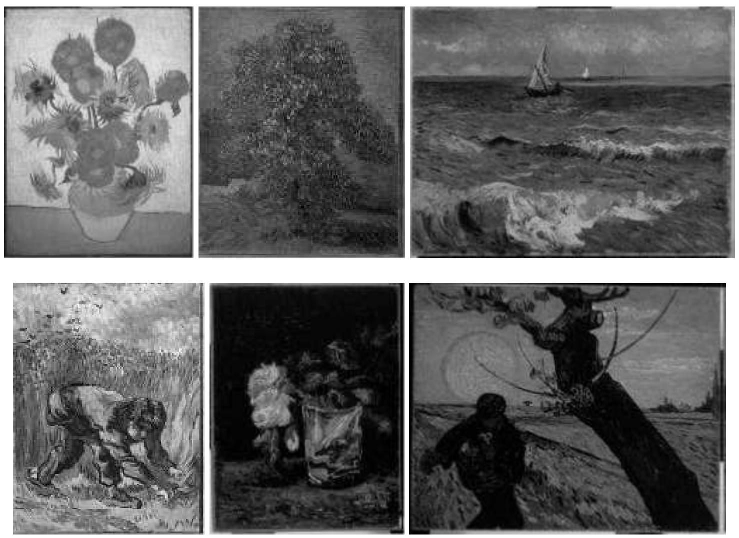

Figure 2: *The 'NOVA challenge' of the Van Gogh Museum: where is the fake Van Gogh?*

It could be supposed the wavelet analysis of high-resolution digitalized copies of the paintings would allow analyzing the characteristics of the paintings on the resolution level that would suit them best. The examination of the distribution of wavelet coefficients would thus describe differences in style and brush strokes. In case of a copy the brush strokes would be painted rather hesitantly, she supposed, and this way she identified the fourth painting (of the child collecting grain in a field) as the copy. It was a hit, but the cheers of her research team would appear to be premature.

4. DENIAL AND CONFIRMATION

Indeed, during the second edition of that same workshop, this time in New York and in collaboration with the MoMA, a more difficult challenge awaited the participants. Now, an additional collection of twenty-one paintings was proposed, while the copy was of the hand of a contemporary of Vincent Van Gogh. Some panic quickly arose: all brush strokes in the twenty-one paintings seemed painted in a hesitating manner, when compared to the six previous paintings. It caused a totally confusion and confronted with this riddle, she eventually asked the organizers of the contest how this could be. Their answer was surprising: the new images were all different simply because they were created with a new scanner. Goodness!

This created an unpleasant suspicion: had she correctly guessed the fake Van Gogh the previous year for wrong reason? In order to conclude this in a definitive way, her team developed a new technique, in which an extremely fine scale study the amount of 'blur' or 'fuzzy spots' in the images was measured by relying on the edges, corners and ends of the lines. A 'blur' index of 0 would stand for a sharp picture, and 1 for an image that in fact is but a blurry spot. For the six paintings in the given order given from the first IP4AI meeting, the wavelet group obtained the following blur indices: 0.65, 0.98, 0.90, 0.10, 0.93, and 0.90. The fake painting clearly had the lowest blur index. This confirmed their Amsterdam hypothesis 'a posteriori' and so the New York IP4AI congress eventually became a successful meeting. Their first hit was unfounded though the team has discovered what was 'different' with the 'The Picker', compared with the other five works: the photo was taken more recently and therefore sharper. However, could they really detect a fake painting?

To further prove the usefulness of the wavelet technique in identifying the originality of paintings it was decided to set up a new validation experiment that would be monitored entirely, without intermediaries and without any additional technical complexities. Charlotte Caspers would make seven pairs of paintings using different canvases, brushes and even styles. In each pair there would be one original (always a still life) and one copy, made afterwards. Both would be painted by Caspers, yet, remarkably, it took her half more time to make the copy than to make the original painting.

It turned out that where Caspers used soft brushes, no positive conclusions could be drawn. But when she had used soft and hard brushes, copies of the originals could be distinguished with an accuracy of 86%! In this way, the 'lucky strike' of the IP4AI conference could be justified: the method was correct after all.

5. AN UNDERLYING VAN GOGH

One of the participants in the interdisciplinary mathematics and art workshops was Joris Dik of the Delft University of Technology (The Netherlands). Together with his Belgian colleague Koen Janssens of the University of Antwerp, he had managed to obtain a sharp image of a drawing of the head of a woman made by Van Gogh, though Vincent had painted another work on top of it two years later: 'Patch of Grass'. It was known a portrait by Van Gogh lay under the painting, because he did reuse old canvassed to save money. Moreover, art historians therefore attached particular importance to this underlying portrait as Van Gogh had written to his brother that he was particularly successful in creating an intended effect in that portrait.

Dik's and Janssens' use of the 'Deutsches Elektronen-Synchrotron' (DESY) in Hamburg (Germany) allowed them to get a better picture than in previous estimates of the drawing lines. In the hidden layer, at .06 to .10 millimeters below the layer painted afterwards, they determined the quantities of paint components such as antimony (an element of the so-called 'Naples yellow'), mercury and arsenic (found in the color vermilion), cobalt, lead, and other chemical components. In this way the lines did not only emerge more clearly than with prior techniques, but also the colors.

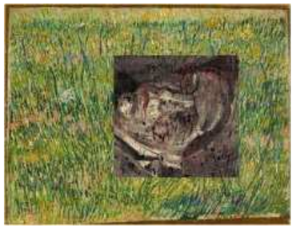

Figure 3: *Van Gogh's 'Patch of Grass' and the underlying portrait.*

There were problems though: where the paint in 'Patch of Grass' was very thick, the analysis based on the 'X-ray fluorescence', could not provide enough data. In that case, there were black spots. This led Dik to suggest the image could be improved using digital technology. Dik's team had made pixel by pixel measurements, and this caused inaccuracies of synchronization causing anomalies in the alignment of parts of the picture. The wavelet team developed a special yet completely different mathematical algorithm. Next, they used coloring techniques to fill in the black spots. Finally, they determined the color even better by relying on the pallet Van Gogh used for other portraits he made in Nuenen. And behold, a lost Van Gogh emerged! Japanese art investors can contact the Academy for more information.

6. GOOSSEN VAN DER WEYDEN

Another encounter led to work of Goswin or Goossen van der Weyden (Brussels approx. 1465 - Antwerp, 1538), the grandson of the more famous Rogier van der Weyden. A sabbatical at the Free University of Brussels and at the VLAC, the Flemish Academic Center, which is division of the Royal Flemish Academy of Belgium, led to contacts with art historian Maximilian Martens (Gent University, Belgium; VLAC), who did research there as well, together with his colleague Marc De Mey. The latter is also involved with experiments about the Mystic Lamb, and Martens' research group coordinates the scientific work at the Gent University during the forthcoming restoration of the painting by the Van Eycks.

It happened one of the conversations was about Goswin van der Weyden, who made several now famous paintings, such as for an Abbey in the small Belgian town of Tongerlo and for the Saint Gummarus church in another city, Lier. His triptych 'Presentation of Jesus in the Temple' even moved to Lisbon, Portugal. Goossen van der Weyden used a great variety of different styles for its underlying drawings. They were not only a consequence of certain periods in his career, sometimes they also included indications for his employee who would eventually paint that part of the work (the master didn't necessarily work alone). The question remained if it could be mathematically verified that an underlying drawing style characteristic consistently corresponded to the part painted above it. That is, can the drawing and painting features really be classified 'mathematically', or are they "the most individual expressions of the most intense personal emotions" as some Dutch poet once lyrically described art?

	Probability in %						
Drawing style ↓	Detail A	Detail B	Detail C	Detail D	Detail E	Detail F	Detail G
1	0	8	4	2	8	1	8
2	3	31	6	13	18	14	42
3	96	28	82	60	53	37	49
4	1	33	8	25	21	48	1
Correct answer	3	3	3	4	1	4	2
Success ?	yes	close	Yes	close	no	Yes	close

Figure 4: *For the woman's head of Goossen van der Weyden (detail E), the mathematical method was not conclusive, because two drawing styles were used simultaneously: style 1 (in green in the drawing above) and style 3 (red).*

The team received a collection of details of paintings with the corresponding under-drawings and their classification. They 'trained' an algorithm to distinguish the corresponding styles of painted surfaces. Next, they got seven fragments of painted details, unknown to them, and they were asked to predict what the underlying style type would be. Math can be exciting, yes indeed.

Their algorithm gave the probabilities at which a detail of the painting corresponded to a given drawing style. Only the analysis for the proposed detail E was disappointing, but a further investigation turned out that two styles were used simultaneously in this singular case. Thus, the technique would not even have been capable to provide a decisive conclusion in this case. The paintings and drawing characteristics by Van der Weyden turned out to be truly measurable, mathematically. They appeared to be more than mere individual expressions of emotions.

7. MYSTIC LAMB

Today, the methods for processing images are applied to Mystic Lamb. The techniques do not only concern wavelets anymore. Indeed, two teams now investigate on the use of image processing in art, one at the University of Ghent, Belgium, with Aleksandra Pižurica, Ljiljana Platiša, and Tijana Ruzic (TELIN-IPI-IBBT Department), and another one at the Free University of Brussels, Belgium, with Bruno Cornelis and Ann Dooms (ETRO-IBBT Department).


Figure 5: *The Mystic Lamb.*

Moreover, interested readers are encouraged to investigate themselves, as they can freely consult very precise images of the Mystic Lamb on the website 'Closer to Van Eyck' (http://closertovaneyck.kikirpa.be/). It shows extremely high resolutions images, for as many as 100 billion pixels of images were processed. And in case of a question, readers can always contact the cheerleader…

PART 2: WORKSHOPS AND EXHIBITIONS HELD AT THE DEPARTMENT OF ARCHITECTURE SINT-LUCAS, ON MAY 11 AND MAY 12